WHAT ARE EIGEN-WERTE?

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1. Introduction. Of the many stages upon which the dramas of mathematical analysis are enacted, a notable one is that upon which the *Eigen-wert* plays a major role. The inception of the plot there centers in the main around the interrelations of mathematics and the physical sciences, and its development shows well how these sister disciplines influence each other—how the association has yielded inspiration and direction to the one, and to the other precision, power and form.

The differential equation is the most powerful mathematical tool for the formulation of physical laws and problems. Hence, when formulated mathematically, a vast category of physical problems call for a solution of some differential equation—in the great majority of cases for such a solution as fulfills on the side certain prescribed "initial conditions" insofar as the time is concerned, and certain "boundary conditions" relative to its space coordinates. It is pertinent to consider in outline, a few such formulations. Though the configurations to be referred to are specialized, and the remarks but brief and fragmentary, they nevertheless serve as vehicles for the ideas we wish to set forth. Throughout the discussion the symbol $\nabla^2\phi$ will be used to signify the expression

$$\frac{\partial^2\phi}{\partial x^2}, \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}, \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2},$$

according as the portion of space that is in question, and which will be denoted by V, is one, two, or three dimensional. The boundary (points, curve or surface) of V will be designated by S, and $\partial\phi/\partial n$ will be interpreted as the normal derivative if S is a curve or surface, and as $\partial\phi/\partial x$ if x is the sole space dimension of V.

2. Eigen-werte in problems of vibration. If a homogeneous and isotropic material body occupies a space V, its elastic vibrations are governable by a function ϕ which satisfies a pair of equations

$$\begin{aligned} &\nabla^2\phi = a^2\frac{\partial^2\phi}{\partial t^2}, \quad \text{within } V, \\ &\alpha\phi+\beta\frac{\partial\phi}{\partial n}=0, \quad \text{upon } S. \end{aligned} \tag{1}$$

In these a^2 is a non-vanishing constant, and α and β are constants, or more generally functions of the space coordinates. At the initial instant which may be taken without loss of generality as $t=0$, the function ϕ fulfills furthermore a pair of relations

$$\begin{aligned} &\phi\Big|_{t=0} = f_1, \\ &\frac{\partial\phi}{\phi t}\Big|_{t=0} = f_2, \end{aligned} \tag{2}$$

in which f_1 and f_2 are prescribed functions.

A classical method for an analysis of such a function ϕ proceeds along the lines of the following considerations. If any product of a factor v, which depends only on the time, by a factor u which is independent of that variable, is to solve the equations (1), that condition is found at once to imply the relation

$$\frac{\nabla^2 u}{a^2 u} = \frac{\dfrac{d^2 v}{dt^2}}{v}.$$

Inasmuch as neither member of the equality depends upon any variable which the other appears to involve, they must both be constant. With the designation of this constant common value by $-\lambda$ (the minus sign is a mere formal convenience), it follows that v and u must respectively be solutions of the equations.

$$\frac{d^2 v}{dt^2} + \lambda v = 0, \tag{3}$$

and of the "differential boundary problem"

$$\begin{aligned} &\nabla^2 u + \lambda a^2 u = 0, \quad \text{within } V, \\ &\alpha u + \beta \frac{\partial u}{\partial n} = 0, \quad \text{upon } S. \end{aligned} \tag{4}$$

The equation (3) is solvable irrespective of the value of λ. Not so the boundary problem (4). This, on the contrary, can be shown by suitable analysis to be solvable by a non-trivial function (*i.e.*, by one that does not vanish identically) only if λ has one of a specifically singled-out set of values. These values, which depend upon the coefficients of the problem and upon the boundary S, are known as "characteristic values," or, in German terms, as *Eigen-werte*. The non-trivial solutions which exist for these values are known as "characteristic solutions," or *Eigen-funktionen*. If u_k is such a solution and λ_k the corresponding characteristic value, the respective solution of the equation (3) is

$$v_k(t) \equiv c_k \cos \rho_k t + \gamma_k \sin \rho_k t, \tag{5}$$

with $\rho_k^2 = \lambda_k$, and, however the constants c_k and γ_k may be chosen, the product $v_k u_k$ satisfies the equations (1).

This bears in the following way upon the envisaged determination of the function ϕ. Because of the linearity of the equations (1), they are satisfied by any finite sum of products $v_k u_k$, and beyond that by any infinite series

$$\sum_{k=1}^{\infty} v_k(t) u_k, \tag{6}$$

which has suitable properties of convergence. These properties may be attainable through suitable choices of the coefficients c_k and γ_k which appear in (5).

If a series (6) is to represent the function ϕ, however, it is furthermore requisite, by virtue of the conditions (2), that the series

$$\sum_{k=1}^{\infty} v_k(0)u_k, \qquad \sum_{k=1}^{\infty} \frac{dv_k}{dt}\bigg|_{t=0} u_k, \tag{7}$$

represent respectively the prescribed functions f_1 and f_2. Whether, and under what conditions, given functions are so representable in infinite series of characteristic solutions, are problems in the so-called "expansion theory" which is associated with any boundary problem (4). This expansion theory is quite inextricably interwoven with elements from the theory of characteristic values.

3. The case of the vibrating string. Concrete and familiar applications of these considerations are called forth by the instances of the vibrating string or the vibrating membrane. The function ϕ in these cases represents the displacement from the position of equilibrium, and the functions f_1 and f_2 describe the initial displacements and velocities. In the case of the string, under tension, fastened at the points $x=0$ and $x=l$, and vibrating in a plane, the equations (1) are explicitly

$$\frac{\partial^2\phi}{\partial x^2} = a^2\frac{\partial^2\phi}{\partial t^2}, \quad \text{for} \quad 0 \leqq x \leqq l, \tag{1*}$$

$$\phi\,|_{x=0} = 0, \qquad \phi\,|_{x=l} = 0.$$

The boundary problem (4) is accordingly

$$\frac{d^2u}{dx^2} + \lambda a^2 u = 0, \tag{4*}$$

$$u(0) = 0, \qquad u(l) = 0,$$

and the simplest considerations suffice to show that any solution of the differential equation in (4*) that satisfies one of the boundary conditions, can satisfy the other also if and only if λ is one of the values $\lambda_k = (k\pi/al)^2$, with $k = 1, 2, 3, \cdots$. The associated characteristic solutions are the functions $\sin k\pi x/l$, and the series (7), which are to represent the functions $f_1(x)$ and $f_2(x)$, are respectively

$$\sum_{k=1}^{\infty} c_k \sin\frac{k\pi x}{l}, \tag{7*}$$

$$\sum_{k=1}^{\infty} \gamma_k \frac{k\pi}{al} \sin\frac{k\pi x}{l}.$$

In this case the expansion theory is evidently the theory of Fourier's series.

4. Eigen-werte in problems of flow and potential. A category of physical problems, of which the flow of heat in a material body may be taken as presenting a typical one, admits of formulation through the medium of a function ϕ

(the temperature in the case of heat flow) which is characterized as the solution of the differential system

$$\nabla^2\phi = a^2\frac{\partial\phi}{\partial t}, \quad \text{within } V,$$

$$\alpha\phi + \beta\frac{\partial\phi}{\partial n} = 0, \quad \text{upon } S, \tag{8}$$

$$\phi\,|_{t=0} = f_1.$$

Although of a physical character quite distinct from those of vibration, these problems admit of analysis differing only in minor details from that above. In particular, the appearance of characteristic values and their rôle, is precisely such as has been indicated.

Many problems, such as those of steady flow, say of electricity through a conductor, are mathematically subsumed in the theory of potential. The governing differential system in such cases is of the form

$$\nabla^2\phi = 0, \quad \text{within } V,$$

$$\phi = f, \quad \text{upon } S, \tag{9}$$

the function f being again prescribed. In the instance of certain configurations V and functions f such problems yield to analysis through the medium of characteristic values. Thus if the volume V is a sphere, and with its center as pole the spherical coordinates in which ρ is radial distance and θ, ψ are latitude and longitude are introduced into the equations (9) to replace the Cartesian (x, y, z), it is found that if v is a function of ρ alone and u is independent of ρ, then

$$\frac{\rho^2\nabla^2(vu)}{vu} = \frac{1}{v}\left[\rho^2\frac{d^2v}{d\rho^2} + 2\rho\frac{dv}{d\rho}\right] + \frac{1}{u}\left[\sec^2\theta\frac{\partial^2u}{\partial\psi^2} + \frac{\partial^2u}{\partial\theta^2} - \tan\theta\frac{\partial u}{\partial\theta}\right].$$

Since the first member on the right of this formula is constant as to (θ, ψ) whereas the second is constant as to ρ, they must both be absolute constants, if the product vu is to satisfy the differential equation of (9). The functions v and u must, therefore, be respectively solutions of the equations

$$\rho^2\frac{d^2v}{d\rho^2} + 2\rho\frac{dv}{d\rho} - \lambda u = 0, \tag{10}$$

and

$$\sec^2\theta\frac{\partial^2u}{\partial\psi^2} + \frac{\partial^2u}{\partial\theta^2} - \tan\theta\frac{\partial u}{\partial\theta} + \lambda u = 0. \tag{11}$$

Our immediate concern is with the differential equation (11). It has singular points where $\theta = \pm\pi/2$, and these in general call forth corresponding singularities in the solutions. To be physically useful, however, the solutions $u(\theta, \psi)$ must be

free from singularities. This requisite and the fact can be shown to be compatible if and only if λ has a value from a certain peculiar set. These values, it need hardly be said, are the characteristic values. The determination of ϕ with the use of them, is again a problem in the related expansion theory.

5. Eigen-werte in quantum mechanics. Characteristic values are to be met with in modern physical theory no less than in those more classical. In the theory of quantum mechanics, for instance, they appear in connection with the wave equation. To illustrate this it suffices to regard the mechanical system which consists of only a single particle of mass m, when the potential energy U is a function of the space coördinates alone. The probability that at any specified time this particle be located within a given portion of space, is taken to be computable from a distribution function ϕ, which is postulated to be a solution of the wave equation

$$\frac{h^2}{8\pi^2 m}\nabla^2\phi - U\phi = \frac{h}{2\pi i}\frac{\partial\phi}{\partial t}. \tag{12}$$

If ϕ is assumed to be the product of two factors v and u, which are respectively free from the space and time variables, the equation (12) constrains these functions to be solutions of the equations

$$\frac{dv}{dt} + \frac{2\pi i W}{h}v = 0,$$

$$\nabla^2 u + \frac{8\pi^2 m}{h^2}\{W - U\}u = 0,$$

in which W is a parameter, such as has heretofore been designated by λ. In the instance of many coefficient functions U the requisite, which is physically inspired, that the solution u be bounded throughout the space, constrains the parameter W to take one of a set of *Eigen-werte*. These are then interpreted as the characteristic total energy values of the system.

6. The boundary problem in the theory of ordinary differential equations. Many physical phenomena (the examples considered can suggest their diversity only very inadequately) thus yield to mathematical formulation as boundary problems, in which a linear differential equation involving a free parameter appears in conjunction with a complement of auxiliary conditions. As might well be expected such problems have been the object of much study, some of this motivated by the applicability of the results, but much of it, on the other hand, inspired by the purely mathematical interest which inheres in them. Insofar as ordinary differential equations are concerned, the type of such problems may be taken as of the form

$$\begin{aligned} L(u) - \lambda u &= 0, \\ W_k(u) &= 0, \qquad k = 1, 2, \cdots, n, \end{aligned} \tag{13}$$

in which

$$L(u) \equiv u^{[n]} + p_1 u^{[n-1]} + \cdots + p_n u,$$

and

$$W_k(u) = \sum_{j=0}^{n-1} \{\alpha_{jk} u^{[j]}(a) + \beta_{jk} u^{[j]}(b)\},$$

the symbol $u^{[j]}$ designating the jth derivative of u. In the commonest cases the coefficients α_{jk} and β_{jk} are constants, and the p_j are suitably differentiable functions of x.

The differential equation of the system (13) considered by itself, admits of n linearly independent solutions $u_1(x, \lambda), \cdots, u_n(x, \lambda)$. In terms of these, every solution is expressible in the manner.

$$u(x, \lambda) = \sum_{r=1}^{n} c_r u_r(x, \lambda), \tag{14}$$

with constant coefficients c_r. The substitution of this expression into the boundary conditions yields the set of relations

$$\sum_{r=1}^{n} c_r W_k(u_r) = 0, \qquad k = 1, 2, \cdots, n,$$

which constitutes a linear algebraic system that must be satisfied by the coefficients c_r if the function (14) is to solve the boundary problem. Since these coefficients may not all be zero, inasmuch as the solution is to be non-trivial, it is evidently requisite that

$$|\, W_i(u_j) \,| = 0, \tag{15}$$

the symbol on the left designating the determinant of which the general element is shown. The equation (15) is known as the "characteristic equation." Its variable is the parameter λ, which is involved through the functions $u_r(x, \lambda)$, and its roots are the characteristic values.

It must suffice here to observe that the mathematical literature includes discussions of many different generalizations of the problem (13) as described. The characteristic values in themselves, inasmuch as they are explicitly determinable only in the very simplest cases, have called forth many deductions relative to their approximate or asymptotic representation. In these studies the notation of vectors and matrices has often been found advantageous. Thus the boundary problem

$$u' = \{\rho P(x) + Q(x)\} u,$$
$$W_1 u(a) + W_2 u(b) = 0,$$

in which the capitals stand for square matrices, in which $\rho^n = \lambda$, and in which u

is a vector, is in scalar terms more explicitly

$$u_i' = \sum_{r=1}^{n} \{\rho p_{ir}(x) + q_{ir}(x)\} u_r,$$

$$\sum_{s=1}^{n} \{w_{1,i,s} u_s(a) + w_{2,i,s} u_s(b)\} = 0, \qquad i = 1, 2, \cdots, n$$

and to this the problem (13) is transformable.

7. Eigen-werte in the theory of integral equations. If the differential system

$$\begin{aligned} L(u) &= f(x), \\ W_k(u) &= 0, \qquad k = 1, 2, \cdots, n, \end{aligned} \tag{16}$$

admits of only the trivial solution in the instance that the function $f(x)$ is identically zero, it may be solved in the instance of all other suitably integrable functions by a formula

$$u(x) = \int_a^b K(x, s) f(s) ds,$$

in which $K(x, s)$ is theoretically constructible, and is known as the Green's function. With the formal replacement of $f(x)$ by λu ,the system (16) reverts to the boundary problem (13), and this latter thus appears, insofar as its solution is concerned, as coextensive with the "integral equation"

$$u(x) - \lambda \int_a^b K(x, s) u(s) ds = 0. \tag{17}$$

Many physical problems thus clearly admit of indirect formulation in the manner (17). The indirect route is, however, by no means the only one. Thus, in the case of the vibrating string, a straightforward application of mechanical principles may be made to lead directly to the equation (17), the specific "kernel" in that case being

$$K(x, s) = \begin{cases} \dfrac{s(l - x)}{lT}, & \text{when} \quad 0 \leqq s \leqq x, \\ \dfrac{x(l - s)}{lT}, & \text{when} \quad x \leqq s \leqq l, \end{cases}$$

in which T denotes the tension. The most direct and elementary means will show in this case that the equation (17) admits of a non-trivial solution only if λ is a characteristic value. This suggests, and it is a fact, that the appearance of characteristic values is an important phenomenon in the theory of integral equations. Their rôle in this theory is completely analogous to that which they fill in the theory of boundary problems in differential form.

8. Eigen-werte in Hilbert spaces. The integral equation serves conveniently here as a point of departure toward a domain of greater abstractness. If the kernel $K(x, s)$ is envisaged as any specific function that is of integrable square the relation

$$\Psi(x) = \int_a^b K(x, s)\psi(s)ds, \tag{18}$$

may be looked upon as associating with any function $\psi(x)$ of the space L_2 (*i.e.* of the aggregate of functions whose squares are integrable) another such function $\Psi(x)$. The relation, therefore, effectively defines a transformation from ψ to Ψ, and this may be indicated concisely by the symbolism

$$T\psi = \Psi. \tag{19}$$

Now with appropriate adjustments the space L_2 is includable in a class of spaces known as "Hilbert spaces," and the relation (19) is then significant of a particular linear transformation in such a space. By abandoning the particularization of the space, and the restriction whereunder the relation (19) is a mere restatement of (18), the symbol T is emancipated, and may be taken to stand for the general linear transformation in abstract Hilbert space. In the theory of such transformations the analogue of the equation (17) is the relation

$$I\psi - \lambda T\psi = 0, \tag{20}$$

in which I stands for the identical transformation under which $I\psi = \psi$, for all elements ψ. If λ is a constant for which the transformation $I - \lambda T$ possesses an inverse, only the trivial solution is admitted by the relation (20). The demand for a non-trivial solution can be met, therefore, only if the transformation in question has no inverse. The values of λ for which that is so are the characteristic values.

The simplest of the Hilbert spaces is that in which the elements are the number sequences $\psi \equiv (x_1, x_2, x_3, \cdots)$, for which the series $\sum_{k=1}^{\infty} |x_k|^2$ are convergent. In this space a transformation T is defined by a system of equations

$$y_r = \sum_{k=1}^{\infty} a_{r,k}x_k, \qquad r = 1, 2, 3, \cdots,$$

with constant coefficients a_{ij}. With this interpretation the relation (20) takes on the form

$$x_r - \lambda \sum_{k=1}^{\infty} a_{r,k}x_k = 0, \qquad r = 1, 2, 3, \cdots, \tag{21}$$

and such a system of equations accordingly has a theory of characteristic values associated with it.

9. The reduction of a quadratic form. To bring this discussion to a close, let the system of algebraic equations

$$x_r - \lambda \sum_{k=1}^{m} a_{r,k} x_k = 0, \qquad r = 1, 2, \cdots, m, \tag{22}$$

which is a finite analogue of the infinite system (21), be briefly considered. These equations present themselves in particular in the determination of relative extrema of the quadratic form

$$\sum_{k=1}^{m} x_k^2 \tag{23}$$

in the face of the auxiliary condition

$$\sum_{i,j=1}^{m} a_{i,j} x_i x_j = 1, \tag{24}$$

the matrix $(a_{i,j})$ being symmetric. The parameter λ appears in this instance as a Lagrange multiplier. Since the form (23) is geometrically interpretable as the square of the distance from the center of the quadric surface (24), the problem is clearly that of locating the end points of the principal axes of that surface. If λ_i is any characteristic value of the system (22), and $x_r = x_r^i$ is the corresponding characteristic solution, the multiplication of the equations by the respective x_r and their summation relative to r, yields by virtue of the condition (24) the evaluation

$$\sum_{r=1}^{m} x_r^{(i)^2} = \lambda_i.$$

This shows the characteristic values to be the squares of the semi-axes. Under an orthogonal transformation which lays the coördinates along these axes, the equation (24) would accordingly be replaced by

$$\sum_{k=1}^{m} \frac{y_k^2}{\lambda_k} = 1.$$

From this it will be evident that the characteristic values are significant in the theory of the normalization of quadratic forms.

The writer will plead limitations of space and the purpose of this note as excuses for the many omissions, which will be all too obvious, and for the cursoriness with which all matters have been touched upon. The subject is a vast one. Nor has its closing chapter yet been written. Its significance will maintain far into the future.

A MAGIC CUBE WITH $6n^3$ CELLS

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An *ordinary magic cube of the nth order* is here defined as an arrangement of n^3 different positive integers at the vertices of an $n \times n \times n$ (three dimensional) lattice work in such a manner that the sum of the numbers in every row, every column, every file, and in each of the four main diagonals* shall be the same number S. When n is greater than three, it is also required that the sum along the diagonals of each horizontal layer shall have the value S. The number S will be called the *characteristic* of the magic cube.

It may be noted in passing that earlier writers on this topic have been content with a definition of a magic cube which imposed no restriction on the sums along the diagonals of the horizontal layers. It seems to the author that the definition above is more appropriate and more interesting. As an indication of the possibilities of further development along the lines which will be suggested by this paper, it may be said that the writer has recently constructed ordinary magic cubes of orders six and ten which conform to the definition in the first paragraph.

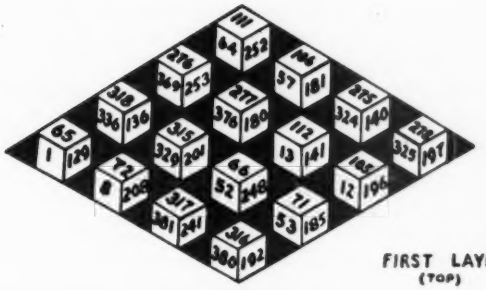
It is the purpose of this paper to explain the construction of a "generalized magic cube" which will now be defined. Consider a cube as made up of n^3 "cubelets" (of equal dimensions) arranged in layers of n^2 cubelets each. On each face of every one of these cublets let there be written one of the integers from 1 to $6n^3$ inclusive, without repetition. Let it be required that the numbers written on the top faces of the cubelets form an ordinary magic cube. Likewise those on the bottom faces, those on the north faces, those on the south faces, those on the east faces, and those on the west faces. (In the four cases last mentioned, it is understood that the $n \times n$ arrays whose diagonals are required to have the sum S lie in vertical planes, instead of the horizontal planes as specified in the first paragraph.) It is also required that all of these ordinary magic cubes have the same characteristic. Such an arrangement of the first $6n^3$ positive integers will be called a generalized magic cube.

A generalized magic cube can be obtained from an ordinary magic cube by multiplying all of the numbers by 6, and then judiciously adding to or subtracting from each of them a suitably chosen number from the set 0, 1, 2, 3, 4, 5, so as to obtain the proper combination of six magic cubes with no duplication of numbers.

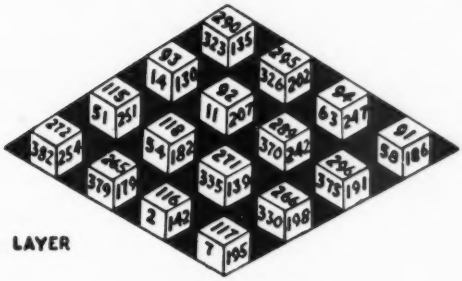
Such a generalized magic cube with $n=4$ was constructed by the author in 1931. In the present paper there is presented a generalized magic cube with $n=4$ which has the additional property that the sum of the integers on each of the cubelets is equal to 1155. The manner in which this result was obtained will now be explained.

First, a magic square was constructed by use of the first sixteen positive

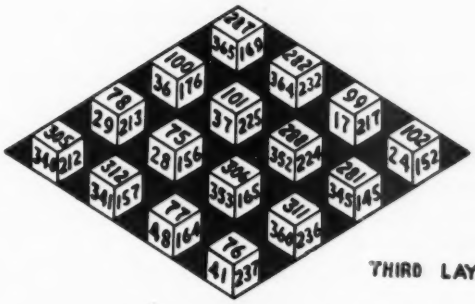
* A main diagonal of the cube is understood to be one of the four lines joining two vertices which do not lie in the same face of the cube.



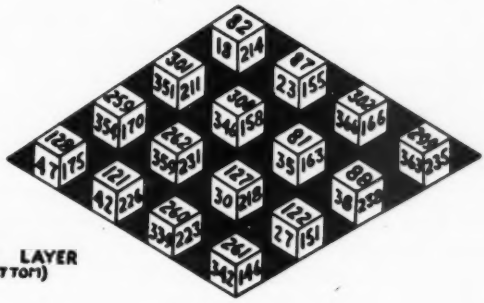
FIRST LAYER
(TOP)



SECOND LAYER



THIRD LAYER



FOURTH LAYER
(BOTTOM)

integers. Then, by a natural extension of the methods which have been developed in connection with magic squares, a magic cube was formed by use of the first sixty-four positive integers. Care was exercised, however, to form the cube in such a way that in each horizontal layer every integer in the first two rows was *complementary* to the corresponding integer in the last two rows. (Two integers are said to be complementary if their sum is equal to the sum of the least and the greatest of the integers employed. In this case, two integers whose sum is 65 are complementary.)

Then, using the first 384 positive integers, six magic cubes were formed from the original one by the method explained in the fourth paragraph of this paper. Each of these cubes was so constructed that the elements were complementary in pairs as explained in the preceding paragraph. (In this connection, two integers whose sum is 385 are complementary.) The next step was to effect a rearrangement so as to obtain two ordinary magic cubes such that each integer in one of them would be the complement of the corresponding integer in the other. This purpose was accomplished as follows. Two of the six cubes were selected, say cubes A and B. Cube A, keeping the first two rows in each of its four layers, had the numbers in the third and fourth rows of each layer replaced by the integers occupying corresponding positions in cube B. Let the new cube so obtained be denoted by C. Then a cube D was formed whose first and second rows in each layer were the third and fourth rows of the corresponding layers of A; and whose third and fourth rows in each layer were the first and second rows of the layers of B. Thus were obtained two magic cubes C and D such that each term in one of them was complementary to the corresponding number in the other.

The same procedure was applied to another pair of the six cubes, and finally to the remaining pair. Thus were obtained three pairs of magic cubes, each cube being complementary, term by term, to the one with which it was paired. It was found that this could be done in such a way that each of the six cubes had the same characteristic.

Finally, the terms of each of the six ordinary magic cubes were inscribed on the faces of the cubelets in such a manner that the integers on opposite sides of the cubelets were in all instances complementary. The figure shows only three faces of each cubelet. But the integers not shown can be obtained by subtracting the visible numbers from 385.

In the generalized magic cube shown in the figure, the sum 770 will be obtained by adding the four integers in any row, column, file, main diagonal, or any one of the previously specified diagonals of squares. The sum of the six numbers on any one of the cubelets is 1155, as can easily be proved by means of the fact that the numbers on opposite faces are complementary. The sum 770 will also be obtained by adding the four numbers visible on any one of the cubelets when any two opposite faces are covered. This fact is obvious since the sum will be $1155-385$. It is easily verified that the sum 770 can be obtained in 552 ways from the figure.

The figure exhibits other uniformities not pointed out here. The reader may enjoy discovering them for himself.

The results obtained in this paper suggest the problem of constructing a cube similar to the one in the figure, but with the property that each of the twenty-four layers is a pan-diagonal magic square; or with the property that each of the six ordinary magic cubes is a pan-diagonal magic cube. In the latter connection see W. W. R. Ball, *Mathematical Recreations and Essays* (1939), page 221.

LEGENDRE FUNCTIONS OF THE SECOND KIND AND RELATED FUNCTIONS*

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1. Introduction. A reviewer of my recent Carus Monograph† has noted as an omission the absence of any reference to Legendre functions of the second kind.‡ The purpose of the present paper is to remedy the omission in some degree by an introductory treatment of these functions which is comparable as to its elementary character with the treatment of the Legendre polynomials in the Monograph, and which in particular makes no use of the theory of functions of a complex variable. It is further pointed out that the same method is applicable to a discussion of second solutions of the differential equations of Hermite, Laguerre, Jacobi, and Bessel.

The Legendre functions of the second kind are met with in the problem of solving completely the Legendre differential equation (formula (10) below), which is of importance in mathematical physics and elsewhere. For each non-negative integral value of n the equation has as one solution the Legendre polynomial of corresponding degree. A second solution is readily found for any particular value of n by a well known standard procedure. From the point of view of a comprehensive theory the properties of the second solutions are in many respects materially less simple than those of the polynomials. There is however one striking resemblance: the familiar and fundamentally important relation of recurrence (formula (1)) connecting any three successive Legendre polynomials is satisfied also by the corresponding second solutions, when the arbitrary constants are suitably chosen. This property appears further deserving of particular emphasis in an introductory account because it makes possible the rapid tabulation of complete solutions of the differential equation for successive values of n in explicit form.

It is natural to inquire how directly it is possible to trace a formal connection between the differential equation itself and the recurrence formula. In

* Presented to the Minnesota Section of the Association at Northfield, Minn. May 9, 1942.

† Fourier Series and Orthogonal Polynomials, Carus Mathematical Monograph No. 6, 1941.

‡ See Review of Scientific Instruments, vol. 13, 1942, pp. 180–181.

particular, if $y_0, y_1, y_2, \cdots$ form a sequence of functions connected by the recurrence relation, and if the initial functions y_0, y_1, which suffice to determine the rest of the sequence, are solutions of the differential equation for the values of n equal to their respective subscripts, can it be inferred from these facts alone that the later functions necessarily satisfy the differential equation for the corresponding values of the parameter? The answer to this question is in the negative; if y_{n-1}, y_n, y_{n+1} are connected by the recurrence formula, and if y_{n-1} and y_n satisfy the differential equations appropriate to their subscripts, the expression

$$(1 - x^2)y''_{n+1} - 2xy'_{n+1} + (n + 1)(n + 2)y_{n+1}$$

reduces, not to zero, but to

$$\frac{2(2n + 1)}{n + 1}[(1 - x^2)y'_n + nxy_n - ny_{n-1}].$$

The quantity in brackets vanishes identically when the y's are the Legendre polynomials of the degrees indicated by the subscripts, or second solutions with the constants suitably determined. Furthermore it can be shown that if this identity holds for $n=1$ and for $n=2$ it must continue to hold by virtue of the recurrence relation. Therefore a process of mathematical induction can be set up for the differential equation in conjunction with this additional identity.

It may be pointed out also that if the first *four* functions y_0, y_1, y_2, y_3 are solutions of the respective differential equations, the recurrence relation holding throughout, then the supplementary identity is necessarily fulfilled for $n=1$ and for $n=2$, and the whole induction goes through accordingly.

It is not the purpose of this paper to ask the reader to follow through the particular process of calculation that has been outlined. When the possibility of the reasoning by mathematical induction has been recognized it becomes apparent that the manipulation of the formulas admits almost endless variation, and there are alternative arrangements of the work which simplify the details appreciably. On comparison of a number of different procedures, one growing naturally out of another, it seems most advantageous to make use of expressions relating y'_n and y_n respectively to the pair of functions y'_{n-1} and y_{n-1} (formulas (2), (3), (8), (9)), and this arrangement is followed below, without further protracted account of the successive stages of experiment leading to the form which is ultimately adopted.

It is perhaps not surprising that the method of treatment which is effective for the Legendre differential equation can be generalized to the equation satisfied by arbitrary Jacobi polynomials, among which those of Legendre are included as a special case, and can be applied also to the differential equations for the closely related orthogonal polynomials of Hermite and Laguerre. The place of these various systems of orthogonal polynomials in a single comprehensive theory is set forth in detail in the Monograph referred to at the beginning of the paper. Especially noteworthy is the fact that the type of treatment under pres-

ent consideration applies likewise to Bessel's equation, which is associated with that of Legendre through its significance in mathematical physics, but is otherwise less similar in its formal theory. Still other applications are omitted from discussion here. The ultimate scope of the method is not superficially apparent.

2. Recurrence formulas and the Legendre differential equation. Let $y_0(x)$, $y_1(x)$, $y_2(x)$, $\cdots$, be a sequence of functions such that

$$(1) \qquad (n+1)y_{n+1}(x) - (2n+1)xy_n(x) + ny_{n-1}(x) = 0, \qquad n = 1, 2, \cdots.$$

Let it be supposed that each of these functions has first and second derivatives over the range of values of x to be considered. Let

$$(2) \qquad u_n(x) = y_n'(x) - ny_{n-1}(x) - xy_{n-1}'(x), \qquad n = 1, 2, \cdots,$$

$$(3) \qquad v_n(x) = ny_n(x) - nxy_{n-1}(x) + (1-x^2)y_{n-1}'(x), \qquad n = 1, 2, \cdots.$$

From (1),

$$(4) \qquad (n+1)y_{n+1} = (2n+1)xy_n - ny_{n-1},$$

and consequently

$$(5) \qquad (n+1)y_{n+1}' = (2n+1)xy_n' + (2n+1)y_n - ny_{n-1}'.$$

Substitution of (5) in the relation

$$(n+1)u_{n+1} = (n+1)y_{n+1}' - (n+1)^2y_n - (n+1)xy_n',$$

obtained from (2) on replacement of n by $n+1$ and multiplication by $n+1$, gives

$$(n+1)u_{n+1} = nxy_n' - n^2y_n - ny_{n-1}'.$$

By combination of (2) and (3), on the other hand,

$$xu_n - v_n = xy_n' - ny_n - y_{n-1}'.$$

Consequently

$$(6) \qquad (n+1)u_{n+1} = n(xu_n - v_n).$$

Similarly, from (3) and then (4),

$$v_{n+1} = nxy_n - ny_{n-1} + (1-x^2)y_n',$$

while the combination $(1-x^2)u_n + xv_n$ is found from (2) and (3) to reduce to the same expression, so that

$$(7) \qquad v_{n+1} = (1-x^2)u_n + xv_n.$$

The relations (6), (7), holding for $n \geqq 1$, show that if $u_1 \equiv v_1 \equiv 0$ then $u_2 \equiv v_2 \equiv 0$, and by induction $u_n \equiv v_n \equiv 0$ for all n. That is to say, *if the y's, connected by* (1), *are such that the identities*

$$y_n' - ny_{n-1} - xy_{n-1}' = 0, \tag{8}$$

$$ny_n - nxy_{n-1} + (1 - x^2)y_{n-1}' = 0 \tag{9}$$

hold for $n=1$, they hold for all positive integral n.

By differentiation of (3) and combination with (2),

$$v_n' - nu_n = (1 - x^2)y_{n-1}'' - 2xy_{n-1}' + (n - 1)ny_{n-1}.$$

The identities (8) and (9), stating that $u_n \equiv v_n \equiv 0$, imply that the right-hand member vanishes for $n \geqq 1$, which is equivalent to saying that

$$(1 - x^2)y_n'' - 2xy_n' + n(n + 1)y_n = 0 \tag{10}$$

for $n \geqq 0$.

The conclusions of the last two paragraphs can be combined to yield the following:

THEOREM I. *If $y_0(x)$, $y_1(x)$ are any two functions (possessing first and second derivatives) such that*

$$y_1' - y_0 - xy_0' = 0, \tag{11}$$

$$y_1 - xy_0 + (1 - x^2)y_0' = 0, \tag{12}$$

and if other functions $y_2(x)$, $y_3(x)$, $\cdots$ are calculated successively by (4), the general function $y_n(x)$ of the sequence thus defined satisfies the Legendre differential equation (10).

It is to be repeated also that under the conditions stated (8) and (9) hold separately, and furthermore, naturally, any other identities that can be deduced from these. For example, since

$$u_{n+1} = y_{n+1}' - (n + 1)y_n - xy_n',$$

$u_{n+1} + xu_n - v_n$ reduces to $y_{n+1}' - (2n+1)y_n - y_{n-1}'$, and as $u_{n+1} \equiv u_n \equiv v_n \equiv 0$,

$$(2n + 1)y_n = y_{n+1}' - y_{n-1}'. \tag{13}$$

3. Legendre polynomials and Legendre functions of the second kind. If the functions $y_0(x) = 1$, $y_1(x) = x$ are substituted in (11) and (12) it is immediately apparent that both identities are satisfied. Calculation of y_2, y_3, $\cdots$, from (4) with these values of y_0 and y_1 for a beginning gives for y_n a polynomial of the nth degree. By Theorem I this polynomial, conventionally denoted by $P_n(x)$, is a solution of (10). The polynomials P_0, P_1, P_2, $\cdots$, are the *Legendre polynomials.** A substantial introduction to their theory is contained in the recurrence formula (1) or (4), the differential equation, and the identities (8), (9), and (13), from which further properties can readily be deduced.

For another sequence of y's let

* See for example the Monograph already cited, Chapter II.

$$y_0 = \tfrac{1}{2} \log \frac{1+x}{1-x},$$

$$y_1 = -1 + \tfrac{1}{2}x \log \frac{1+x}{1-x} = -1 + xy_0.$$

These also are found to satisfy (11) and (12). The variable x is to be restricted now to the interval $-1<x<1$, so that the logarithm shall be real. As indication of the source of these functions, let it be noted that the differential equation

$$(1-x^2)y'' - 2xy' = 0,$$

to which (10) reduces for $n=0$, can be solved by the most elementary means, being an equation of the first order for y':

$$\frac{y''}{y'} = \frac{2x}{1-x^2}, \qquad \log y' = c - \log (1-x^2), \qquad y' = \frac{k_1}{1-x^2},$$

$$y = k_1 \int \frac{dx}{1-x^2} = k_1 \cdot \tfrac{1}{2} \log \frac{1+x}{1-x} + k_2,$$

so that the general solution is a linear combination of the particular solution 1 used in the preceding paragraph and the present y_0. If then y_1 is defined in terms of this y_0 by means of (12), the form obtained for y_1 is that given above; and (11) is found to be satisfied also.

By a different arrangement of the work the whole sequence of y's can be built up from the single function y_0 by successive applications of (9), and the accompanying theory developed accordingly, without separate special consideration of y_1. For any sequence of functions $y_0, y_1, y_2, \cdots$, possessing first and second derivatives, but not assumed at the outset to satisfy (1), let v_n and u_n have the meanings assigned to them by (3) and (2), and let w_n denote the left member of (10). Then

$$u_n = (v_n' - w_{n-1})/n,$$

$$w_n = (1-x^2)u_n' - 2xu_n + (n+1)v_n + xw_{n-1},$$

as may be verified by evaluation of the right-hand members. From these identities it follows that if y_0 is a function for which $w_0 \equiv 0$, and if $y_1, y_2, \cdots$ are defined successively by setting $v_n \equiv 0$ for $n=1, 2, \cdots$, then $u_1 \equiv 0$, $w_1 \equiv 0$, and by further induction $u_n \equiv 0$ and then $w_n \equiv 0$, that is, (10) is satisfied, for all positive integral n. Also

$$(n+1)y_{n+1} - (2n+1)xy_n + ny_{n-1} = v_{n+1} - xv_n - (1-x^2)u_n,$$

so that the identical vanishing of all the u's and v's implies the relation (1). This scheme of operations is perhaps simpler in plan than that followed in the earlier paragraphs, but the actual verification in detail appears to be more laborious.

Let $Q_n(x)$ denote the general function y_n of the sequence defined by (4) with

the logarithmic y_0 and y_1 of the second paragraph preceding. It has the form $\phi_n(x)y_0+\psi_n(x)$, in which $\phi_n(x)$ and $\psi_n(x)$ separately satisfy the recurrence relation

$$(n+1)\phi_{n+1} = (2n+1)x\phi_n - n\phi_{n-1},$$
$$(n+1)\psi_{n+1} = (2n+1)x\psi_n - n\psi_{n-1}.$$

For recognition of the last fact it suffices to substitute $\phi_n y_0+\psi_n$ and $\phi_{n-1}y_0+\psi_{n-1}$ in (4) for y_n and y_{n-1}. Since $\phi_0=1=P_0(x)$ and $\phi_1=x=P_1(x)$, the ϕ's are identical with the Legendre polynomials, while the initial determinations $\psi_0=0$, $\psi_1=-1$ make ψ_n a polynomial of degree $n-1$, which for the sake of putting its degree explicitly in evidence may be denoted alternatively by $R_{n-1}(x)$. Then*

$$Q_n(x) = \tfrac{1}{2}P_n(x)\log\frac{1+x}{1-x} + R_{n-1}(x).$$

By Theorem I, the function $y_n=Q_n(x)$ satisfies (10). As it is clearly not a constant multiple of $P_n(x)$, it is a second solution independent of $P_n(x)$, and the general solution of (10) is of the form

$$AP_n(x) + BQ_n(x).$$

The Q's are *Legendre functions of the second kind.*

Any particular Q_n can also be calculated without the necessity of building up from Q_0 and Q_1 through the recurrence formula if the corresponding P_n is known, by assuming a solution of (10) in the form

$$y_n = \tfrac{1}{2}P_n(x)\log\frac{1+x}{1-x} + z_n,$$

with z_n as a new unknown function. Then

$$y_n' = \tfrac{1}{2}P_n'(x)\log\frac{1+x}{1-x} + \frac{P_n(x)}{1-x^2} + z_n',$$

$$y_n'' = \tfrac{1}{2}P_n''(x)\log\frac{1+x}{1-x} + \frac{2P_n'(x)}{1-x^2} + \frac{2xP_n(x)}{(1-x^2)^2} + z_n'',$$

and by virtue of the fact that $P_n(x)$ satisfies (10)

$$(1-x^2)y_n'' - 2xy_n' + n(n+1)y_n = (1-x^2)z_n'' - 2xz_n' + n(n+1)z_n + 2P_n'(x).$$

If a polynomial of degree $n-1$ is substituted for z_n in the right-hand member, its coefficients can be uniquely determined in descending order so that the expression shall reduce to zero.

* For another method of deriving this form of solution, and for further particulars concerning the polynomials $R_{n-1}(x)$, see E. W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Cambridge, 1931, pp. 49–51, 53–55.

The work of the preceding section shows further that the Q's satisfy the relations (8), (9) and (13).*

4. Associated functions. The differential equation

$$\frac{d^2T}{d\theta^2} + \cot\theta \frac{dT}{d\theta} + \left[n(n+1) - \frac{k^2}{\sin^2\theta}\right] T = 0 \tag{14}$$

is met with in connection with the solution of Laplace's equation in spherical coordinates.† By the substitution $x = \cos\theta$ this becomes

$$(1-x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx} + \left[n(n+1) - \frac{k^2}{1-x^2}\right] T = 0. \tag{15}$$

A further change of dependent variable, defined by the relation $T = (1-x^2)^{k/2}z$, gives for z the differential equation

$$(1-x^2)\frac{d^2z}{dx^2} - 2(k+1)x\frac{dz}{dx} + [n(n+1) - k(k+1)]z = 0. \tag{16}$$

On the other hand, the result of differentiating (10) k times in succession is

$$(1-x^2)y^{(k+2)} - 2(k+1)xy^{(k+1)} + [n(n+1) - k(k+1)]y^{(k)} = 0,$$

which is the same as (16) if $y^{(k)}$ is identified with z. That is to say, if y is any solution of (10), $z = y^{(k)}$ satisfies (16). In particular, $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ are solutions of (16).

Consequently (15) has for solutions the *associated Legendre functions*

$$(1-x^2)^{k/2}P_n^{(k)}(x), \qquad (1-x^2)^{k/2}Q_n^{(k)}(x),$$

and corresponding solutions of (14) are

$$\sin^k\theta P_n^{(k)}(\cos\theta), \qquad \sin^k\theta Q_n^{(k)}(\cos\theta).$$

5. The Hermite differential equation. In modification of the initial hypotheses of §2 let $y_0, y_1, y_2, \cdots$ now be a sequence of functions satisfying the recurrence relation

$$y_{n+1} - xy_n + ny_{n-1} = 0, \qquad n \geqq 1. \tag{17}$$

In place of (2) and (3) let

$$u_n = y_n' - ny_{n-1}, \qquad n \geqq 1,$$

$$v_n = y_n - xy_{n-1} + y_{n-1}', \qquad n \geqq 1.$$

* For additional information about the functions $Q_n(x)$ reference may be made for example to Hobson, *op. cit.*, especially pp. 49–72, and Whittaker and Watson, A Course of Modern Analysis, Cambridge, 1920, and other editions, Chapter XV.

† See Monograph, pp. 118–120.

Formulas (4) and (5) are replaced by

$$y_{n+1} = xy_n - ny_{n-1},$$
$$y'_{n+1} = xy'_n + y_n - ny'_{n-1}. \tag{18}$$

Further calculation corresponding to that of §2 gives

$$u_{n+1} = y'_{n+1} - (n+1)y_n = xy'_n - ny_n - ny'_{n-1} = xu_n - nv_n, \tag{19}$$

$$v_{n+1} = y_{n+1} - xy_n + y'_n = y'_n - ny_{n-1} = u_n, \tag{20}$$

$$v'_n - u_n = y''_{n-1} - xy'_{n-1} + (n-1)y_{n-1}. \tag{21}$$

The recurrence relation expressed by (17) or (18) being assumed to hold throughout, (19) and (20) show that the identical vanishing of u_1 and v_1 implies that of u_n and v_n for all positive integral n, and (21), holding for $n \geqq 1$, means then that

$$y''_n - xy'_n + ny_n = 0 \tag{22}$$

for $n \geqq 0$. Thus the following analogue of Theorem I is obtained:

THEOREM II. *If $y_0(x)$, $y_1(x)$ are any two (twice differentiable) functions such that*

$$y'_1 = y_0, \tag{23}$$

$$y_1 = xy_0 - y'_0, \tag{24}$$

and if other functions $y_2(x)$, $y_3(x)$, $\cdots$ are calculated successively by (18), *the general function $y_n(x)$ of the sequence thus defined satisfies the Hermite differential equation* (22).

Under the same hypotheses, *the y's satisfy also the relations*

$$y'_n = ny_{n-1}, \qquad y_n = xy_{n-1} - y'_{n-1}, \tag{25}$$

which express the vanishing of u_n and v_n.

If $y_0 \equiv 1$, $y_1 \equiv x$, satisfying (23) and (24), the function $y_n(x)$, which is seen from (18) to be a polynomial of the nth degree, is the *Hermite polynomial** $H_n(x)$. It is a solution of (22), by Theorem II; and successive H's satisfy the relations (25).

A second particular solution of (22) for $n = 0$ is obtained by writing:

$$y''_0/y'_0 = x, \qquad \log y'_0 = x^2/2, \qquad y'_0 = e^{x^2/2},$$

$$y_0 = \int_0^x e^{x^2/2}dx.$$

With this y_0 let y_1 be defined by means of (24) as

$$y_1 = x\int_0^x e^{x^2/2}dx - e^{x^2/2}.$$

* See Monograph, Chapter IX.

It is apparent that (23) is satisfied also. Further definition of $y_2, y_3, \cdots$ by means of (18) gives for y_n an expression of the form

$$y_n = H_n(x) \int_0^x e^{x^2/2} dx + r_{n-1}(x) e^{x^2/2},$$

where $H_n(x)$ is the Hermite polynomial, and $r_{n-1}(x)$ is a polynomial of degree $n-1$. Let this y_n be denoted by $G_n(x)$. It is obviously not a constant multiple of $H_n(x)$, if the fact that $\int e^{x^2/2} dx$ is not a function of "elementary" form is assumed as known; in any case the conclusion is assured by the observation that the G's satisfy the relations (25), from the first of which it follows by repeated application that if G_n were a polynomial of the nth degree, G_{n-1} would have to be a polynomial of degree $n-1$, and ultimately G_0 would have to be a constant, which it certainly is not. The general solution of (22) is of the form

$$AH_n(x) + BG_n(x).$$

6. The Laguerre differential equation. For the relations (4), (2), (3) let the following now be substituted:

$$\begin{aligned} y_{n+1} &= (x - \alpha - 2n - 1)y_n - n(\alpha + n)y_{n-1}, \\ u_n &= y_n' - ny_{n-1} + ny_{n-1}', \\ v_n &= y_n - (x - \alpha - n)y_{n-1} + xy_{n-1}'. \end{aligned} \tag{26}$$

Straightforward carrying out of the indicated operations verifies the identities

$$\begin{aligned} u_{n+1} &= (x - \alpha - n)u_n - nv_n, \\ v_{n+1} &= xu_n - nv_n, \\ v_n' - u_n &= xy_{n-1}'' + (\alpha + 1 - x)y_{n-1}' + (n - 1)y_{n-1}. \end{aligned}$$

From these may be read off, in analogy with Theorems I and II:

THEOREM III. *If $y_0(x)$, $y_1(x)$ are any two (twice differentiable) functions such that $u_1 \equiv v_1 \equiv 0$, that is to say, such that*

$$y_1' = y_0 - y_0', \tag{27}$$

$$y_1 = (x - \alpha - 1)y_0 - xy_0', \tag{28}$$

and if other functions $y_2(x), y_3(x), \cdots$ are calculated successively by (26), the general function $y_n(x)$ of the sequence thus defined satisfies the Laguerre differential equation

$$xy_n'' + (\alpha + 1 - x)y_n' + ny_n = 0. \tag{29}$$

Successive y's are connected by the relations

$$y_n' = ny_{n-1} - ny_{n-1}', \tag{30}$$

$$y_n = (x - \alpha - n)y_{n-1} - xy_{n-1}'. \tag{31}$$

If $y_0=1$, $y_1=x-\alpha-1$ (the form of y_1 being prescribed by (28) when y_0 is given), both (27) and (28) are satisfied. The corresponding y_n is the *Laguerre polynomial** $L_n(x)$ or $L_n^{(\alpha)}(x)$ of the nth degree, for the value of α which appears in (29). The polynomials of the sequence thus obtained satisfy (29), (30), and (31).

For $n=0$, a non-constant solution of (29) is given by the formulas

$$\frac{y_0''}{y_0'} = 1 - \frac{\alpha+1}{x}, \qquad \log y_0' = x - (\alpha+1)\log x,$$

$$y_0 = \int_1^x \frac{e^x dx}{x^{\alpha+1}};$$

any other positive number would serve equally well instead of 1 as lower limit of integration. The corresponding y_1 from (28) is

$$y_1 = (x-\alpha-1)\int_1^x \frac{e^x dx}{x^{\alpha+1}} - \frac{e^x}{x^\alpha},$$

and this function together with y_0 satisfies (27). The general function y_n of the sequence now defined by (26) is of the form

$$y_n = L_n(x)\int_1^x \frac{e^x dx}{x^{\alpha+1}} + \rho_{n-1}(x)\frac{e^x}{x^\alpha},$$

in which $L_n(x)$ and $\rho_{n-1}(x)$ are the Laguerre polynomial and a polynomial of degree $n-1$ respectively. If this $y_n=M_n(x)$ were a polynomial it would follow from the identity

$$xy_n' - ny_n = n(\alpha+n)y_{n-1},$$

which is obtained from (30) and (31) by elimination of y_{n-1}', that $y_{n-1}, \cdots, y_0$ must be polynomials, in contradiction with the definition of y_0. Consequently $M_n(x)$ as a solution of (29) is independent of $L_n(x)$, and the general solution is of the form

$$AL_n(x) + BM_n(x).$$

7. The Jacobi differential equation. For a corresponding treatment of the general Jacobi polynomials and the accompanying second solutions, the key formulas are the differential equation and the recurrence formula on pages 174 and 173 of the Monograph, and the combinations

$$u_n = 2(\alpha+\beta+n)y_n' - (\alpha+\beta+n)(\alpha+\beta+2n)y_{n-1} - [(\alpha+\beta+2n)x + (\beta-\alpha)]y_{n-1}',$$

$$v_n = 2n(\alpha+\beta+n)y_n - (\alpha+\beta+n)[(\alpha+\beta+2n)x + (\alpha-\beta)]y_{n-1} + (\alpha+\beta+2n)(1-x^2)y_{n-1}'.$$

* See Monograph, Chapter X.

The calculations are naturally somewhat laborious, and will not be given here at length. With the understanding that $\alpha > -1$, $\beta > -1$, the reader who has the patience to verify the identities

$$2(n+1)(\alpha+\beta+2n)u_{n+1} = (\alpha+\beta+2n+2)\{(\alpha+\beta+2n)(xu_n - v_n) + (\beta-\alpha)u_n\},$$

$$2(\alpha+\beta+n)(\alpha+\beta+2n)v_{n+1} = (\alpha+\beta+2n+2)\{(\alpha+\beta+2n)[(1-x^2)u_n + xv_n] + (\beta-\alpha)v_n\}$$

will have little difficulty with the rest. In making comparisons with §2 it is to be noted that if the expressions defined by (2) and (3) are denoted now by s_n and t_n, the present u_n and v_n reduce for $\alpha=\beta=0$ not to s_n and t_n, but to $2ns_n$ and $2nt_n$ respectively, while u_{n+1} and v_{n+1} become $2(n+1)s_{n+1}$ and $2(n+1)t_{n+1}$.

Exceptional circumstances arise if $\alpha+\beta+1=0$, so that the factor $\alpha+\beta+n$ vanishes for $n=1$. In this case the differential equation has for $n=1$ the non-polynomial solution $y_1=(1-x)^{-\alpha}(1+x)^{-\beta}$. A corresponding solution for $n=2$ is obtained from this y_1 by setting $v_2=0$, and solutions $y_3, y_4, \cdots$ for higher values of n are then given by the recurrence formula.

8. The Bessel differential equation. For Bessel functions the formulas are comparatively simple once more, except for the less elementary definition of the initial functions in the sequences. The relations to be considered are those satisfied by the functions $J_n(x)$ for non-negative integral n and the functions of the second kind associated with them.

Let $y_0, y_1, y_2, \cdots$ be a sequence of functions connected by the recurrence formula

$$y_{n+1} = (2n/x)y_n - y_{n-1} \tag{32}$$

for $n \geqq 1$. Let

$$u_n = y_n' - \left(1 - \frac{n(n-1)}{x^2}\right)y_{n-1} - \frac{n}{x}y_{n-1}', \qquad n \geqq 1,$$

$$v_n = y_n - \frac{n-1}{x}y_{n-1} + y_{n-1}', \qquad n \geqq 1.$$

Then

$$u_{n+1} = \frac{n-1}{x}u_n - \left(1 - \frac{n(n-1)}{x^2}\right)v_n, \tag{33}$$

$$v_{n+1} = u_n + \frac{n}{x}v_n, \tag{34}$$

$$v_n' - u_n = y_{n-1}'' + \frac{1}{x}y_{n-1}' + \left(1 - \frac{(n-1)^2}{x^2}\right)y_{n-1}, \tag{35}$$

for the same values of n.

Let y_0 be any function satisfying the differential equation

$$y_0'' + (1/x)y_0' + y_0 = 0, \tag{36}$$

which is obtained by setting the right-hand member of (35) equal to zero with $n=1$. Let y_1 be defined by making $v_1=0$, with this choice of y_0. The whole sequence of y's is then determined. It follows from (35) that $u_1=0$, and then by an induction based on (33) and (34) that $u_n=v_n=0$ for all $n\geqq 1$. Consequently the right-hand member of (35) vanishes for each value of n, which means, on replacement of $n-1$ by n, that y_n is a solution of the Bessel differential equation

$$y_n'' + \frac{1}{x}y_n' + \left(1 - \frac{n^2}{x^2}\right)y_n = 0. \tag{37}$$

If $y_0=J_0(x)$, setting $v_1=0$ makes $y_1=-J_0'(x)=J_1(x)$, and it is seen that $y_n=J_n(x)$ for each n, since this is true for $n=0$, 1, and the y's are connected by the recurrence relation which is known to be satisfied by the functions $J_n(x)$ as otherwise defined.*

Now let y_0 be instead the function

$$y_0 = J_0(x)\log x + \frac{x^2}{2^2} - (1+\tfrac{1}{2})\frac{x^4}{2^2 4^2} + \cdots$$

which is a second solution† of (36). The relation $v_1=y_1+y_0'=0$ gives

$$y_1 = -J_0'(x)\log x - (1/x)J_0(x) - \tfrac{1}{2}x + \cdots,$$

or, since $-J_0'(x)=J_1(x)$, while $J_0(x)$ has the form $1-x^2/2^2+\cdots$,

$$y_1 = J_1(x)\log x - (1/x)\pi_1(x),$$

in which $\pi_1(x)$ is a power series in non-negative powers of x with constant term unity. It is found by application of (32) for the evaluation of $y_2, y_3, \cdots$ that

$$y_n = J_n(x)\log x - 2^{n-1}(n-1)!x^{-n}\pi_n(x),$$

where $\pi_n(x)$ is in each case a power series in non-negative powers of x with 1 as constant term. This y_n is a second solution of (37).

It will be observed that the results which have been presented in this and the preceding sections do not by any means exhaust the generality of the method employed. For example, the sequence of indices 0, 1, 2, $\cdots$ may with appropriate modifications of statement be replaced by a non-integral sequence proceeding at unit intervals; and in §§6 and 7 the parameters α and β need not be restricted, as in the ordinary theory of orthogonal polynomials, to values greater than -1, though the admission of values outside this range in §7 requires further attention to exceptional cases.

* See Monograph, p. 86.

† See Monograph, p. 219.

MATHEMATICS INSTRUCTION IN ENGINEERING

J. W. CELL, North Carolina State College

1. Introduction. As chairman of three different committees of the Mathematics Division of the Society for the Promotion of Engineering Education, the writer of this article has been engaged for the past four years in collecting engineering problems which illustrate mathematics and which are suitable for use in teaching freshman and sophomore courses in engineering mathematics. The Mathematics Division has entered into a non-royalty agreement with the McGraw-Hill Book Company for the publication of the resulting collection of 510 problems which will be available in book form this spring.

The actual work of collecting these problems and a survey of the final collection have suggested many ideas about mathematics instruction. It is the purpose of these notes to point out some of them. The writer has recently engaged in teaching mathematics for advanced radio training, and that experience also may color the following remarks. But in order to avoid any possible misunderstanding, the writer does not intend his remarks to suggest or to imply that the teaching of engineering mathematics should be solely utilitarian, or that we should teach the engineering student only such mathematics as he will need in his junior and senior engineering courses. It does stand to reason, however, that if certain required instruction in engineering mathematics would bring out the same general mathematical abilities as does some of the traditional mathematical subject matter, then it would be well to consider teaching the former in place of the traditional material. On the other hand, if a change in teaching methods should be suggested by engineering requirements, and if this change could be shown to indicate improvement in the mathematics instruction itself, then the teacher of mathematics would profit by such suggestions. It is in the light of these two possibilities that the following remarks are to be read. It is certain, however, that some mathematical topics are necessary for the proper understanding of mathematics itself, irrespective of possible later applications.

2. College algebra. The topic of completing the square in college algebra is an example of a mathematical procedure which has little direct engineering application. We teach it for its technique value and as the means of proof of the quadratic formula. Incidentally, we should correlate it with later mathematical needs as exemplified by the two following problems:

1. Complete the square under the radical $\sqrt{3-4x-2x^2}$ and give your result as the square root of a sum or difference of two quantities.
2. Complete the square on both x and y in the equation

$$2x^2 + 4y^2 - 4x - 12y = 7.$$

It is almost a truism that if a quadratic equation occurring in an engineering problem is solvable by the method of factoring, then the student has made an error.

The topic of radical equations likewise has certain manipulative values but

few actual direct applications in engineering. Further, it presents the first opportunity to discuss multiple-valued functions and the choice of a principal value. Here, too, we should introduce subsequent needs in analytic geometry in the form of problems, such as the elimination of the radicals from the equation

$$\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=10.$$

A third topic the value of which is chiefly in the development of algebraic skill is the topic of equations in quadratic form. While I was gathering the problems for this collection I observed many applications to engineering of the general process of introducing a letter for a combination of letters or for a function of a given variable.

The investigation of engineering texts convinced me that most of the simultaneous linear equations found in engineering should be solved by methods of addition and subtraction or of substitution and not by the use of determinants. Where the coefficients are decimal numbers, some check method should be used in the process of addition and subtraction, such as the check on the algebraic sum of all coefficients and the constant term. Applications for determinants were found in proving theorems, in generating equations whose solutions would ensure non-trivial solutions of a set of linear homogeneous equations, in solving simultaneous linear equations with literal coefficients, and in testing the character of the roots of a given rational integral equation in one variable. The applications involved the theory of determinants rather than the use of determinants for solving simultaneous equations with numerical coefficients.

Most problems in engineering courses are word problems. According to engineering educators, one of the fundamental abilities to be achieved by the time of graduation from college is a thorough understanding of the engineering method and elementary competence in its use. The engineering or scientific method for the solution of problems comprises the four steps:

1. Sketching figures, such as free-body diagrams; labeling all relevant points, lines, etc.;

2. Determining what fundamental principles apply (Newton's laws of motion, Ohm's law, basic theorems from plane geometry, etc.); applying the principles and obtaining a mathematical problem;

3. Solving the mathematical problem;

4. Discussing the engineering implications of the mathematical results, the limitations that were originally imposed, and their consequences.

In the first quarter of the freshman year we teachers of engineering mathematics should initiate the development of this ability by our teaching of word problems.

The collection contains few higher-degree equations requiring solution by Horner's method, for in engineering problems the data seldom warrant accuracy to more than three significant figures, and so it is customary to solve higher-degree equations by cut-and-try methods and by graph. The approximate solution by slide rule of a cubic equation in one variable is easy if the second-degree

term is first eliminated by a process taught in the chapter on the theory of equations. In some cases where complex roots are required, as in electrical and in aeronautical engineering, Graeffe's method or some other approximate method is used. It seems clear that the reasons for teaching this chapter on the theory of equations, as well as the chapter on determinants, lie in the illumination of general theory, possible transformations, and manipulative technique.

3. Trigonometry. Word problems in trigonometry require adequate figures and serve as a second opportunity to teach the translation of them from English to mathematics and the general use of the engineering method. Incidentally, the mathematics teacher should distinguish carefully between word problems which need only common sense as the guide to sketching the required figures, and the other word problems which entail later scientific concepts as aids in figure sketching.

The accuracy of the given data in the majority of engineering problems is such that a solution by the slide rule is sufficient. It would seem appropriate, therefore, to introduce instruction in the slide rule early in the freshman year and to require the slide-rule solution of many problems in trigonometry. At North Carolina State College, at the specific request of the engineering college, we have introduced instruction in the slide rule into the course in trigonometry. We give rules for accuracy simultaneously and expect the student to choose between the slide rule and tables. We have found that the use of the slide rule increases the effectiveness of subsequent teaching, creates active interest, and does not necessitate the sacrifice of other topics in the course. One great advantage of the use of the slide rule is that we can drop the artificial numbers and special angles so common in analytic geometry and calculus problems and dispel the illusion that any involved result is automatically incorrect.

The uses of complex numbers in electrical engineering, especially in polar and exponential form, are many, as an inspection of any text on alternating currents will show. The collection of problems contains many of these applications.

The chapter on multiple angles has many applications, both direct and indirect. It was an amazing experience to me to learn that the really important multiple-angle formulas for the electrical engineer are those which transform a product to a sum or difference.

4. Analytic geometry. Locus derivation problems in analytic geometry serve as the third opportunity to teach the engineering method. Indeed, the steps required in a locus derivation are basically those of this method.

An examination of the problems on conics will show that most of the applications concern either sketching or finding the equation from simple data, and that there are few problems that require information about the foci or eccentricity. An equation from an engineering problem that requires the sketch of some one of the conics often has decimal coefficients, so that the use of the latus rectum is laborious and inferior to the method of finding the coordinates of a few well-chosen points.

The increasing use of transcendental, parametric, and polar coordinate graphs in engineering suggests that we should devote more time to their study. For example, the 2.3 relation between logarithms with the base e and logarithms with the base 10 is easily remembered after it has been visualized on a graph. Further, modern radio texts make use of a graphical solution for one parametric graph $y=f(t)$ from the graphs of $g(x, y)=0$ and $x=h(t)$. The solution consists in considering all three graphs as three views of a space curve and in using engineering drawing methods to complete the solution. I have learned that a somewhat similar approach makes easy the graphical construction of $g(x, y)=0$ (to use the preceding notation). Rapid sketching methods may be used to sketch the two parametric graphs, and then the required graph is easily completed.

Engineering teachers would be eternally grateful if instructors in mathematics taught the students a little about log log and semilog graph paper, since much of the modern engineering data are plotted directly on one of these two types of paper. (Note the relation to the question of decimal versus significant-figure accuracy.) One place to introduce this topic is in empirical equations in analytic geometry.

5. Calculus. One of the most recurrent suggestions in the correspondence about the collection of problems was that we devote more time to graphical and numerical differentiation and integration. Many engineering problems require the derivative or integral curve when the original function is given as a graph. I am convinced, especially by my experiences in this advanced radio training, that the teaching of mathematics would be much more effective if we placed greater emphasis on graphical instruction. Sophomore calculus students should be required to construct accurate, undistorted graphs for several standard functions such as $y=\sin x$ and $y=e^x$, and to determine both the derivative and integral curves by graphical and numerical methods.

Maximum-minimum and related-rate problems in the differential calculus serve as a fourth opportunity to teach the engineering method as well as mathematical methods. Some engineering applications require the largest or smallest value of a function in a given interval, which implies that the student must determine the end values as well as the intermediate extremum values.

The applications of integral calculus will show that there are few uses of volumes, lengths of arc, etc. In junior and senior engineering texts many definite integrals are obtained by the same general methods that we use to obtain the definite integrals in integral calculus. The chief application of these topics, then, consists in the art of setting up definite integrals and not in the use of the formulas in heavy print to be found in most calculus texts. (Sometimes I wish these formulas in heavy print were to be found only in the footnotes.) Although the solution of the usual calculus problems would be longer if the students were required to sketch and label the figure and typical element, to set up the differential statement for that typical element, to sum for all of the elements to obtain a definite integral, and finally to perform the integration; nevertheless this sequence of steps is indicated by engineering needs.

Whether we like it or not we teachers of mathematics must admit that most junior and senior engineering students are going to use integral tables for all but the simplest integrals. It would seem proper, therefore, for us to devote at least one assignment in integral calculus to the proper use of these tables.

6. Multiple-step problems. One ability that is especially difficult for the student to acquire is the identification of the successive mathematical steps required in a multiple-step problem. If the instructor has told him to solve a quadratic equation by the quadratic formula, the procedure is obvious. But if the engineering problem is to determine the positions of zero horizontal stress in a built-in horizontal beam, a subsequently derived quadratic equation may be quite mysterious. Again, the calculus problem of finding the area under the curve $y=\sin x$ from $x=1$ to $x=3$ would cause difficulty because the student would probably not remember that the angles are in radian measure. While single-step problems are necessary in each assignment, we should give at least one problem in each assignment that requires the use of some step learned in a preceding course, the use of which is not indicated in the problem. Mathematics texts should have many such problems, not only for their value as review problems but also for the development of ability in identification.

7. Conclusion. The difficult elements in the engineering method are the translation from engineering to mathematics and the determination of the required mathematical steps. Starting with word problems in college algebra and continuing through setting up definite integrals in calculus, we in mathematics have the opportunity to inaugurate the development of the engineering method. Most of the suggestions in this paper relative to methods of instruction are substantiated by problems in this S.P.E.E. collection. A few comments result from fruitless search for illustrative problems, whether for inclusion in this collection or for use in my own teaching.

APPLIED MATHEMATICS*

H. B. PHILLIPS, Massachusetts Institute of Technology

This paper is a brief report concerning the development of work in applied mathematics at the Massachusetts Institute of Technology. Applications of mathematics had always been emphasized at M.I.T. but plans to cover the various applied fields systematically were first seriously considered in 1934. At that time it was decided to have in the mathematics department at least one specialist in each important applied field, these specialists to be responsible for the program of education and research in their various fields. In particular it seemed desirable to include industrial statistics, analytical statistics, mechani-

* Presented at the twenty-fifth summer meeting of the Mathematical Association of America, at Poughkeepsie, New York, September 7, 1942.

cal applications, electrical applications, and fluid dynamics. An effort was accordingly made to find specialists in these fields.

Unfortunately this was during the depression and so far as we were concerned conditions were becoming steadily worse. Our only opportunity of obtaining new men was therefore through the choice of instructors to replace professors reaching retirement age. In some cases these instructors had little previous training in the fields they were expected to represent but seemed to have the right qualities and were willing to try.

The problem of industrial statistics is principally one of inspection and quality control in mass production industries. It soon became evident that this is as much a problem of economics as of mathematics. Arrangements were therefore made to have the work handled in cooperation with the economics department. Courses, seminars, and industrial surveys were conducted jointly by members of the two departments. After a few years problems in industry were investigated, first as a part of the educational program and then on a contract basis for pay to the Institute. At the outbreak of the war these contracts showed prospect of supporting our entire program in applied mathematics and at the same time supplying problem work in an important field.

Work in analytical statistics was much more easily provided since this had long been one of Professor Wiener's principal fields. The most interesting problem is that of extrapolation in time series. When relations exist it is a question of carrying these forward. The methods first used for doing this were too complicated but these methods have recently been simplified and are now becoming part of the general statistical technique.

In the electrical field it was recognized that mathematicians work at a particular disadvantage due to electrical engineers being mathematically the best trained of all the engineers. For success in competition with them it is necessary to use mathematical methods which they have not sufficiently exploited. Of such methods integral equations seemed to have best prospect of success. The principal difficulty of using integral equations is in reducing results to numerical form. As his first task our representative in this field, Professor Crout, therefore developed methods for obtaining numerical solutions of integral equations. This was accomplished by use of polynomial approximations somewhat analogous to Simpson's Rule. These methods were then used on problems first for practice and then on work handled under contract for particular firms.

Mathematical applications to mechanics consist mainly in the determination of the strength and vibration characteristics of suggested structures. As a basis for this, graduate courses in mechanics and theory of elasticity were given in the department of mathematics. The principal applications made so far have been in the field of aeronautics.

In the meantime work of a very mathematical character was also being done in the science and engineering departments. In 1941 arrangements were, therefore, made to consolidate the mathematical resources of all departments in a research center designed to handle graduate work and research in applied mathe-

matics. This center is under the charge of an interdepartmental committee consisting of representatives of the mathematics department and of all departments in which important applications occur. Graduate students in applied mathematics work directly under the committee, the only limitation being that their study programs include the essentials of one or more fields of application and the mathematics needed to analyze those fields. This type of interdepartmental cooperation has been found particularly satisfactory. Members of each department, being busy with their own affairs, give attention only to matters in which they are particularly interested with the result that each detail is handled by those most competent to handle it.

Funds have been provided for fellowships in applied mathematics but the demand for this type of training is so great that men holding these fellowships have usually been employed in research at the end of one term. The further development of this work is thus even in peace time limited almost entirely by the supply of properly qualified graduate students.

SOME HISTORICAL NOTES ON THE CYCLOID*

E. A. WHITMAN, Carnegie Institute of Technology

1. Introduction. In this paper our interest is not in a renowned mathematician, a celebrated school, or a famous problem, but in a curve, the cycloid. More particularly, our interest is to center around its relation to the mathematics of the seventeenth century, one of the great centuries in the history of the subject. This curve had the good fortune to appear at a time when mathematics was being developed very rapidly and perhaps mathematicians were fortunate that so useful a curve appeared at this time. A new and powerful tool for the study of curves was furnished by the analytic geometry, whose year of birth is commonly given as 1637. New methods for finding tangents to curves, the areas under curves, and the volumes of solids bounded by curved surfaces were being discovered at a rapid pace, and a new subject, the calculus, was in the making. In these developments the cycloid was the one curve used preeminently and nearly every mathematician of the time used it in a trial of some of his new theory, even to the extent that much of the early histories of analytic geometry, calculus, and the cyloid are closely interwoven.

In the history that follows we shall not be concerned with historical minutiae, but only with the broad outlines of the story of this curve.

2. Early history of the curve. The original discoverer of the cycloid appears to be unknown. Paul Tannery has discussed a passage by Iamblichus referring to double movement and has remarked that it is difficult to see how the cycloid

* Presented to the Allegheny Section of the Mathematical Association of America, October 25, 1941.

could have escaped the notice of the ancients.* John Wallis in a letter of 1679, ascribed the discovery to Nicolas Cusanus in 1450 and also mentioned Bouelles as one who in 1500 advanced the study of this curve. In the case of Cusanus, however, historians are agreed that Wallis was mistaken unless, says Cantor, he had access to some manuscript now lost. Now Bouelles mentions that he had observed a rolling wheel yet he seems to have considered the generated arch as a part of a circle whose radius was five-fourths that of the generating circle. The history of the cycloid becomes more definite when we come to Galileo. This scientist and teacher, famed for his telescope and microscope and as the discoverer of the isochronism of the vibrations of a pendulum, this Galileo attempted the quadrature of a cycloidal arch in 1599, at least so writes his pupil Torricelli in a publication of 1644. We here learn that Galileo had sought to measure its area and for this purpose used a balance upon which he placed a material cycloidal arch and a generating circle of like material. Always the arch was about three times as heavy as the circle, wherefore Galileo had given up his experiment since he believed that an incommensurable ratio was in question. Cantor writes of Galileo that he was the first to make this curve well known and that it was he who gave it its name. The curve was also known as a roulette and as a trochoid.

3. The work of Roberval. The scene now shifts to France, to the activities of Gilles Persone de Roberval, and to the problem of the quadrature of the cycloid. Going up to Paris in 1628, Roberval soon became a member of that small group of scientists and mathematicians who were wont to gather twice a week, generally at the home of Père Marin Mersenne, to discuss matters of common interest. Now Mersenne had brought the cycloid to the attention of French mathematicians at various times and Roberval soon learned of this curve but could not immediately effect the quadrature. However, a new method of finding the areas under curves was made known in 1629 when Cavalieri submitted his notes on the theory of *indivisibles* to show his fitness for the chair of mathematics at the University of Bologna, where he was a candidate. This new theory, and its extensions later, exerted an enormous influence upon the subject of finding the areas under curves, hence on the development of the calculus. In this paper we are concerned only with one part of this theory which is known as Cavalieri's Theorem† and which says that if two areas are everywhere of the same width one to the other, then the areas are equal.

About 1634 Roberval effected the quadrature of the cycloid, or trochoid as he called this curve. The first publication of his proof seems to have been in 1693 when his *Traité des Indivisibles*‡ appeared. To explain the long delay in

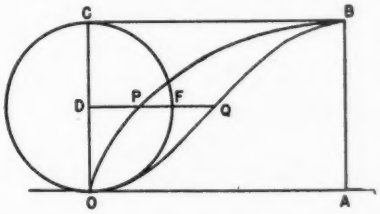
* Pour l'histoire des lignes et surfaces courbes dans l'antiquité; Bulletin des sciences mathématique, Paris, 1883, p. 284.

† A translation of this theorem and its proof is given in Smith's Source Book in Mathematics, also by Professor G. W. Evans in the American Mathematical Monthly for December, 1917.

‡ A Study of Roberval's Traité des Indivisibles, Columbia University, 1932, by Professor Evelyn Walker, gives an extended account of Roberval's works and discusses at some length the subject of indivisibles.

publication of this important discovery, it may be noted that the Chair of Ramus at the Collège Royale which Roberval had won in 1634, automatically became vacant every three years, to be filled again by open competition. As the incumbent set the questions it seems plausible that Roberval should conceal his methods. In this way he would have a set of questions whereby he should win the coming contests. Professor Walker states that the accident of occupying this chair caused Roberval to lose credit for many of his discoveries.

Roberval's quadrature depends upon a so-called cycloid companion curve and an application of Cavalieri's Theorem. Professor Walker gives a translation of this quadrature, but we shall describe it only in a general way. This is among the very earliest of the quadratures.



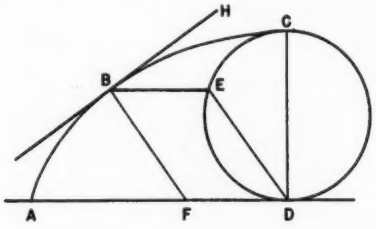
Let $OABP$ be the area under the half arch of the cycloid whose generating circle has the diameter OC. Take P any point on the cycloid and take PQ equal to DF. The locus of Q will be the companion curve to the cycloid. This curve OQB is the sine curve $y = a \sin x/a$ where a is the radius of the generating circle, if we take the origin at the midpoint of the arc OQB, and the x-axis parallel to OA. Now by Cavalieri's Theorem, the curve OQB divides the rectangle $OABC$ into two equal parts since to each line as DQ in $OQBC$, there corresponds an equal line in $OABQ$. The rectangle $OABC$ has its base and altitude equal respectively to the semicircumference and diameter of the generating circle, hence its area is twice that of the circle. Thus $OABQ$ has the same area as the generating circle. Also the area between the cycloid OPB and the curve OQB is equal to the area of the semicircle OFC since these two areas are everywhere of the same width one to the other. Hence the area under the half arch is one and one-half times the area of the generating circle, and the area under the arch is three times that of the generating circle.

4. Construction of the tangent. Early in 1638, Mersenne wrote to Fermat and Descartes presenting for their consideration the problem of the quadrature of the cycloid and the construction of a tangent to the curve. For a year or more previously Roberval and Fermat had been in correspondence, with Senator Carcavy as intermediary. The subjects discussed included tangents, cubatures, and centers of gravity. Mersenne's letters, however, brought to a focus the question of tangents for in August of this year Roberval, Fermat, and Descartes each gave Mersenne a method of drawing a tangent and each had a different method. In the ensuing dispute between Fermat and Descartes over the relative merits of their constructions, Roberval sided with Fermat. In turn Descartes wrote

several letters to Mersenne bitterly ridiculing some of Roberval's tangent constructions which Mersenne had transmitted to him.

The question of priority in the matter of tangents we leave as unsettled and also unimportant, since each could not have borrowed from the others, so different were the methods. Part of the dispute over the relative merits of the constructions arose from different ideas as to the meaning of tangents to curves other than circles. The definition of a tangent as the limiting position of a secant had not yet been generally accepted.

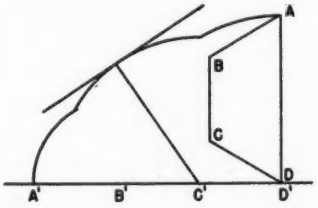
We proceed to describe each of the three tangent constructions. Descartes' method is that which we now call instantaneous centers of curvature.



Let B be any point on the half arch of the cycloid ABC and let it be required to draw a tangent to the cycloid at B.

Draw BE parallel to the base AD cutting the circle at E. Draw BF parallel to ED and BH perpendicular to BF. BH is the required tangent.

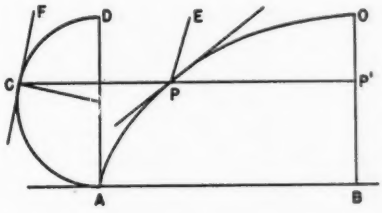
The proof is based on the following considerations:



If a polygon $ABCD$ rolls on a straight line $A'D'$, any point A will describe a number of segments of circles whose centers will be at B', C', D', etc. The tangents to these segments will always be perpendicular to the line joining the point of tangency to the center of the circle. Consequently if the generating circle is considered as a polygon which has an infinite number of sides, the tangent at a given point will be the line perpendicular to the line joining this point to the point where the generating circle touches the base at the same instant it passes through the point.

Roberval's tangent construction makes use of the composition of forces and is easily understood in connection with his particular way of stating the definition of the cycloid.

Let the diameter AD of the circle move always parallel to its original position with A on the line AB until it takes the position BO with AB equal to a semicircumference. At the same time let the point A move on the semicircle ACD in such a way that the speed of AD along AB may be equal to the speed of A along the semicircle, thus allowing A to reach the point D at the same time AD reaches BO. The point A is carried along by two motions, its own on the semicircle and that of the diameter AD. The path of A due to these two motions is the half cycloid APO.



To construct a tangent at any point P on the cycloid, draw PP' parallel to AB cutting the semicircle at C. Then draw CF tangent to the semicircle and draw PE parallel to CF. The bisector of the angle EPP' is the required tangent since it is the resultant of two equal motions.

While finding the two components may be difficult for many curves, yet the cycloid is said to be the eleventh curve for which Roberval thus found tangents.

Fermat's construction is not unlike that of Descartes, but the proof appears to the casual reader to be quite as complicated as that of Descartes is simple. In the course of the proof a straight line is replaced by the arc of a circle. This is equivalent to assuming that an arc of a circle approaches coincidence with a certain straight line, making the method essentially one of limits. To one interested in the early approaches to the calculus, Fermat's method will be more interesting than that of Roberval or Descartes. The methods of the latter show what can be done in special cases and without the calculus. As Fermat's proof is quite long and is readily available elsewhere,* it will not be shown here.

With the area under the cycloidal arch and the tangent construction well mastered by his fellow Frenchmen, Mersenne announced these results to Galileo in 1638. Galileo, now old and blind, passed them on to his pupils Torricelli and Viviana, adding the suggestion that this curve would give a graceful form for the arch of the bridge that was projected for the nearby Arno River at Pisa. These pupils responded with a quadrature and a tangent. The interest thus kindled led Torricelli to a considerable study of the curve. In 1644 he made public his quadrature and a method of drawing a tangent. This was the earliest printed article on the cycloid.

* See Cantor's Vorlesungen über Geschichte der Mathematik, Volume II, pages 861–863, or Walker's Traité des Indivisibles, pages 132–134.

Roberval was angered at seeing another print proofs that he considered his own discoveries. He wrote a letter to Torricelli charging plagiarism. More specifically, Roberval charged that a certain Frenchman had written out Fermat's method of maxima and minima and Roberval's propositions on the cycloid, that these papers had come into Torricelli's hands after the death of Galileo, and that Torricelli had published them as his own. This dispute was cut short by Torricelli's early death in 1647, a death caused, according to Cajori, by this charge of plagiarism.

5. Pascal's mathematical contest. Our next episode in this history centers around Blaise Pascal, known for his *Pensées* and his *Lettres Provinciales* as well as for his mathematical works. After a brilliant early career in mathematics he had turned to theology. But suddenly the old mathematical propensity reasserted itself. Ball writes that Pascal was suffering from sleeplessness and toothache when the idea of an essay on the cycloid occurred to him. To his surprise the tooth ceased to ache. Regarding this as a divine intimation to proceed with the problem, he worked incessantly at it for eight days and completed a tolerably full account of the geometry of the cycloid. As certain questions about this curve had never been publicly answered, a prize was now offered by Pascal under the nom de plume of Amos Dettonville.

The year was 1658 when Newton was sixteen years old. The prizes were two in number, forty and twenty Spanish doubloons. The time allotted was June first to October first. Senator Carcavy was made recipient of the solutions offered and he, Pascal, and Roberval were the judges. The problems were as follows:

(1) The area and the center of gravity of that part of a cycloidal arch above a line parallel to the base.

(2) The volume and center of gravity of the volume generated when the above area is revolved about its base and also about its axis of symmetry.

(3) The center of gravity of the solids formed when each body is cut by a plane parallel to its axis of revolution.

Only two contestants, Wallis and Lalouvère, had submitted offerings when time was called. Ball says that Wallis did not submit solutions for the centers of gravity, and Cajori says that Wallis made many mistakes. Both historians agree that Lalouvère was quite unequal to the task. The judges declared that neither contestant was entitled to a prize.

At the time of this contest, Sir Christopher Wren sent to Pascal his proposition on the rectification of the cycloid, not, however, including any proof. When Pascal showed this to Roberval the latter is said to have proved the proposition immediately, claiming to have known it for many years. To Wren goes the credit for the first publication and its proof,* when Wallis published it as Wren's a year later in his *Tractatus duo*.

While the contest was on, Pascal published his *L'Histoire de la Roulette* and after the decision of the judges, his solution of the problems. With Pascal's and

* The general character of the proof is given in Cantor, Volume II, page 904.

Wallis's publications at this time, the problems of quadrature, tangents, rectification, cubature, and centers of gravity are substantially completed in so far as the cycloid, or roulette as it was better known to the Frenchmen, was concerned. All this was accomplished in a period of about twenty-five years and before Newton's work in the calculus. The principle of *indivisibles*, or *infinites*, or whatever they had used, had in the hands of Roberval, Fermat, Torricelli, Wren, and Wallis led to important results. The cycloid curve was always being used; it was the pre-eminent curve, and its importance was to be seen later.

6. The brachistochrone problem. In another fifteen years, Huygens was using the cyloidal pendulum in an attempt to get a better chronometer and made use of the property that the evolute of the cycloid is another equal cycloid. This same Huygens discovered that a heavy particle reached the bottom of an inverted cycloidal arch in the same length of time no matter from what point on the arch it began its descent. In 1686, Leibniz wrote the equation for the curve, thus showing the rapid progress that was being made in analytic geometry. This equation is given here as Leibniz wrote it since his form shows interesting variations from those employed at present:

$$y = \sqrt{2x - xx} + \int dx/\sqrt{2x - xx}.$$

In the decade following the publication of this equation, the Bernoulli brothers, Jacques and Jean, published several articles on the cycloid. But we shall hurry on to one final episode in the history of the curve.

In June, 1696, Jean Bernoulli proposed a new problem which mathematicians were invited to solve: If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from the point A to the point B in the briefest time. In later amplifying the problem Bernoulli says to choose such a curve that if the curve is replaced by a thin curve or groove and a small sphere placed in it and released, then this sphere will pass from one point to the other in the shortest possible time. Thus the famous brachistochrone problem appeared on the scene. The solution is the inverted cycloidal arch. An elaborate model of the brachistochrone formed a considerable part of the mathematics exhibit at the Golden Gate International Exposition in 1940, from which we may conclude that there is still considerable interest in the problem.

In giving out his solution,* Jean Bernoulli wrote that a new kind of maxima and minima is required. In this solution we see that mathematics had advanced at this time as far as the calculus of variations. In a few more years there began to appear articles on general methods for determining the nature of curves formed by other rolling circles and on curves of descent under activating forces other than gravity. As the cycloid thus loses its pre-eminence, this seems a proper place to close this recital of its history.

* See Smith's Source Book in Mathematics, pages 644–655.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

ANNUAL REPORTS

Some annual club reports for the year 1942–43 have already been received. The editor will be glad to receive the others as soon as possible, so that plans can be made for the ensuing year. Reports should consist of a single paragraph giving a summary of the activities of the club and ending with the names of the officers. A good length is about one double spaced typewritten page. Printed programs and other material of interest, such as bibliographies for club talks, should be included with the club report whenever possible. The best student speaker for the year might be encouraged to write up his talk and submit it to the editor for possible publication.

CLUB REPORTS 1941–42

Mathematics Club, University of Dayton

Semi-monthly meetings were held at which members of the club presented papers: *The extraction of roots of numbers* by Robert Stacy, *Impossibilities and unsolved problems* by Joseph Overwein, *Non-Euclidean geometries* by William Fitzgibbon, *The Hatchet planimeter* by Harold Raybould, *Probabilities* by Robert Mantey, *Maps and charts* by Paul Herking. Three members of the faculty also presented papers: *Nomographs* by Mr. C. G. Peckham, *Operations with a slide rule* by Professor A. R. Weber, *Life and work of Niels Henrik Abel* by Professor K. C. Schraut. In December the Club sponsored a *Symposium on Applied Mathematics* at which Mr. W. E. Restemeyer of the University of Cincinnati spoke on *The Laplace transformation*, Mr. C. A. Ludeke of the University of Cincinnati spoke on *The mathematical and physical conditions causing resonance*, and Professor K. C. Schraut of the University of Dayton spoke on *A method of approximating the sum of a convergent series*. After the Symposium a testimonial dinner was held at the Dayton Engineers' Club in honor of the speakers. Toward the close of the year a steak fry was held at Hills and Dales Park. At the final meeting of the year the guest speaker was Professor C. N. Moore, Director of Graduate Studies at the University of Cincinnati, who delivered a very interesting lecture on *The relationship between pure and applied mathematics*. At this meeting the Annual Dean of Science Book Award was conferred upon William Fitzgibbon for having delivered before the Club the most interesting paper of the year. The book chosen for this year was *Men of Mathematics* by E. T. Bell. At the close of the year the Alumni of the Club established the *Mathematics Club Alumni Award for Excellence in Advanced Mathematics* which is to be conferred each year at commencement upon that senior who, in the opinion of the Faculty of the Department of Mathematics, has distinguished himself most in courses beyond the calculus. The award was conferred upon Robert Stacy. The officers of the Club were: President, Robert Stacy; Vice-President, Joseph Overwein; Secretary, William Fitzgibbon. Professor K. C. Schraut served as Faculty Adviser.

Pi Mu Epsilon, Oklahoma A. and M. College

Twenty new members were initiated at a banquet held April 9, 1942, at which time two prizes were awarded, one to the outstanding freshman in mathematics, and one to the best initiate, both selected by competitive examination. The printed menu for the banquet announced a variety of mathematical solids including *ellipsoids of carbohydrates*, *latticed solids*, and a volume bounded by surfaces whose equations suggested a piece of pie; followed by an after dinner program of *Harmonics*

by Julia Herman, *Differentiation* by Ramez Saab, *Integration* by George Goddard, and *Logic in non-mathematical fields* by Edward Robinson. Other papers presented at club meetings during the year were: *Ciphers and cryptograms* by Peter Messenger, *Mathematics in war* by Lt. Col. Yost of the U. S. Army, *Finite differences* by Dr. Diamond, and *Astronomical distances* by Dr. Mindenhall. The officers for 1942–43 are: Director, Patrick Butler; Vice-Director, William Schierman; Secretary-Treasurer, Lawrence Hanna; Corresponding Secretary, E. F. Allen.

Delta Rho, Southern Illinois Normal University

Interesting lectures were presented at every meeting of the club, mostly by members of the club. Among these were *Books of mathematics* by Vernon Snead, *Adding and subtracting logarithms* by Bill Fisher, and *History of geometry* by Betty Johnson. In February every member of the club cooperated in scoring the tests given for the Mathematics Field Day by the Southern Illinois Council of Mathematics Teachers. A Founder's Day banquet was held in March at which Claude Pyle, an outstanding senior chosen by the department, spoke on *Vector analysis*. A six page *Mathematics News* sheet, printed in March, included many items about the *Delta Rho* honorary fraternity, whose student membership is limited to mathematics majors with averages of at least B in mathematics and to sophomores enrolled for their third term of calculus with at last one A in the calculus. This publication also included news and pictures of mathematics classes in several nearby high schools, and articles written by students about the importance of mathematics in the war effort. The officers of *Delta Rho* for the year were: President, Robert Clendenin; Vice-President, Claude Pyle; Secretary, Mary Downen; Treasurer, William Hentze; Program Chairman, Eugene Ulrich.

Kappa Mu Epsilon, Illinois State Normal University

The activities of the chapter were recorded in a mimeographed booklet dedicated to the alumni of *Kappa Mu Epsilon* who are in the armed forces of the United States. At a homecoming breakfast thirty-two members heard a talk on *The bright and dark side of teaching mathematics* by an alumna, Mildred Schulze. Book reviews by Nancy Hightower and Harold Gambrel were given at the next meeting, and in December the members presented a play *The evolution of numbers*, written by the late Professor H. E. Slaught of the University of Chicago. In January a panel discussion was held on the subject *Student teachers in mathematics*, and in March a radio program entitled *Mathematics for victory* was broadcast over *WJBC* by the following cast of members: Gene Weed, Mildred Bauer, Warren Buck, Don Reeves, Melvin Meisinger. At the last regular meeting in April, Dr. C. N. Mills spoke on *The future in mathematics*, discussing in detail methods of discovering prime numbers. Commanding Officers were: Sponsor, Edith I. Atkin; President Gauss, Shirley Isaacson; Vice-President Pascal, Nancy Hightower; Recording Secretary Ahmes, Geneva Meers; Treasurer Napier, Leo Montgomery; Historian Cajori, Warren Buck; Social Chairman Lilivati, Dorothy Johnson; Corresponding Secretary Descartes, C. N. Mills.

Pi Mu Epsilon, University of Wisconsin

A talk on the subject of *Gyroscopes* was given at the opening meeting by Professor Rudolph Langer. The later meetings included talks on *Methods of apportionment in Congress* by Churchill Eisenhart, *Vector diagrams in air navigation* by Mr. Hillis, *Planimeters* by Walter Sivley, *The use of small scale research models in engineering* by Robert McBurney, *The Gamma function* by Professor R. D. James, and *The arithmetic of infinite numbers* by Mr. Tompkins. A three hour examination consisting of thirteen mathematical problems designed to test the intelligence and resourcefulness of the undergraduate student was sponsored by the chapter, with a first prize of fifteen dollars and a second prize of five dollars. Henry Rogers, son of a professor of French at the University, was winner of the first prize. The officers elected for 1942–43 are as follows: President, Arne Larson; Vice-President, Anne Braun; Secretary, Betty Lohr; Treasurer, Henry Rogers. The corresponding secretary for the past year was Keith LeRoy Clark.

Mathematics Club, University of Kansas

Meetings were held on the first and third Thursdays of each month at 4:45 P.M. following a half-hour period of refreshments. The following topics were announced in advance on a printed program: *Mathematics in Coast Artillery* by Lieutenant Baker, C.A.C.; *Three famous problems of ancient mathematicians* by Jean Bartz, *Some problems in synthetic geometry* by Clark Moots, *A history of the development of algebraic symbols* by Harwood Kolsky, *Gambling* by Howard Barnett, *Computing machines* by John Yarnell, *Summation of some interesting numerical sines* by William Luby, University of Kansas City, *The mechanics of rocket flight* by Howard Gadberry, *The special theory of relativity* by John Ise, Jr., *Mathematics in Economics* by Louise Polson. Bibliographies for some of these topics were prepared and have appeared in the September 1942 issue of this MONTHLY. Officers of the club were: President, Merle DeMoss; Vice-President, Harwood Kolsky; Secretary-Treasurer, Jean Bartz; Social Chairman, Irene McClune; Faculty Adviser, Dean Gilbert Ulmer.

The Square Circle Woman's College, University of North Carolina

A lesson in the mathematics used in the army, taught by Lieutenant E. S. Brown, proved to be the highlight of our programs this year. Another interesting program was the demonstration of the slide rule, varieties of graph paper, an instrument for drawing the ellipse, and a model to represent the fourth dimension. The instrument and model were made by students of the Charlotte, N. C., high school. The traditional ceremony was used for the initiation program. Mathematical games and puzzles, including mathematical bingo, were played at the Christmas party. The last meeting was a picnic; new officers were elected. Officers for 1941–42 were: President, Mary Lou Mackie; First Vice-President (Program Chairman), Janice Pickard; Second Vice-President (Assistant Program Chairman), Zabelle Corwin; Secretary-Treasurer, Shirley Elliot.

The Archimedeans, Winthrop College

Three talks on the lives and works of great mathematicians were presented at the October meeting: *Thales* by Sarah Parks; *Ptolemy* by Lillie Belle Evans; *Desargues* by Jessie Cockfield. Other program topics were *Mathematical logic* by Callie Hartley, *Mathematical recreations* by Frances Pinckney and *Paper folding* by Margaret Rickman. Officers for 1942–43 were chosen as follows: President, Lillie Belle Evans; Vice-President, Kathlyn Bomar; Secretary, Jessie Cockfield; Treasurer, Nellie McGill; Social Chairman, Frankie Cole.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON THE DIVERGENCE OF A SERIES

RICHARD BELLMAN, University of Wisconsin

The purpose of this note is to add another elementary proof of the well-known theorem that the series whose elements are the reciprocals of the primes diverges. This proof is very close to the proof of Hermite, elegant and elementary, that the number of primes is infinite.

Assume $\sum_p 1/p < \infty$. Then there would exist a P such that

$$\sum_{p \geqq P} \frac{1}{p} = \alpha < 1.$$

Hence

$$\left(\sum_{p \geqq P} \frac{1}{p}\right)^2 = \sum_{p,q \geqq P} \frac{1}{pq} = \alpha^2,$$

where p, q, range over all primes $\geqq P$, and similarly

$$\left(\sum_{p \geqq P} \frac{1}{p}\right)^n = \sum_{p,q,\cdots \geqq P} \frac{1}{pq \cdots} = \alpha^n.$$

Hence, since a double series of positive terms can be summed in any manner, $\sum 1/N$, where N ranges over all numbers representable as a product of powers of primes greater than or equal to P, is convergent by comparison with $\sum_1^\infty \alpha^n$.

But

$$\left(\Sigma \frac{1}{N}\right) \frac{1}{\left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{p}\right)},$$

where p is the greatest prime less than P, is equal to

$$\left(\Sigma \frac{1}{N}\right)\left(\Sigma \frac{1}{M}\right),$$

where N has the same connotation as before, and M ranges over all numbers representable as a product of powers of primes less than P. From the unique factorization theorem

$$\left(1 + \Sigma \frac{1}{N}\right)\left(\Sigma \frac{1}{M}\right) = \sum_1^\infty \frac{1}{n}$$

as the product of two convergent series of positive terms and with permissible rearrangements. But $\sum_1^\infty 1/n$ diverges. Hence there is a contradiction, and $\sum_p 1/p$ diverges.

AN ANALOG OF PASCAL'S ARITHMETICAL TRIANGLE

S. F. Bibb, Illinois Institute of Technology

The purpose of this note is to express the coefficients of

$$(1) \qquad y_n = \sum_{s=0}^{r} (-1)^s {}_nD_s x^{n-2s},^*$$

* For another form of the coefficients of this equation see Elementary Theory of Equations, first edition, by L. E. Dickson, p. 83.

$(r=n/2$ if n is even and $(n-1)/2$ if n is odd),

obtained by eliminating the parameter t from

$$x = t + t^{-1}, \qquad y_n = t^n + t^{-n}, \qquad (t \neq 0,\ n = 1, 2, 3, \cdots), \tag{2}$$

as determinants, and to show how they may be arranged as a triangle analogous to that of Pascal, thus:

n	Absolute value of coefficients of x					
1	1					
2	1	2				
3	1	3				
4	1	4	2			
5	1	5	5			
6	1	6	9	2		
7	1	7	14	7		
8	1	8	20	16	2	
9	1	9	27	30	9	
...	...	...	...	...	...	
$n-2$	${}_{n-2}D_0$	${}_{n-2}D_1$	${}_{n-2}D_2$	${}_{n-2}D_3$	${}_{n-2}D_4 \cdots$	${}_{n-2}D_r$
$n-1$	${}_{n-1}D_0$	${}_{n-1}D_1$	${}_{n-1}D_2$	${}_{n-1}D_3$	${}_{n-1}D_4 \cdots$	${}_{n-1}D_r$
n	${}_nD_0$	${}_nD_1$	${}_nD_2$	${}_nD_3$	${}_nD_4 \cdots$	${}_nD_r$

Except for the first element in each row and the last element when n is even, the law of formation of this triangle is given by the formula,

$${}_nD_s = {}_{n-1}D_s + {}_{n-2}D_{s-1}.$$

The D-symbols are defined in the following way:

$${}_nD_0 = 1, \qquad {}_nD_1 = {}_nC_1,$$

$${}_nD_2 = \begin{vmatrix} {}_nC_1 & {}_nC_2 \\ 1 & {}_{n-2}C_1 \end{vmatrix}, \qquad {}_nD_3 = \begin{vmatrix} {}_nC_1 & {}_nC_2 & {}_nC_3 \\ 1 & {}_{n-2}C_1 & {}_{n-2}C_2 \\ 0 & 1 & {}_{n-4}C_1 \end{vmatrix}, \cdots,$$

$$(3) \qquad {}_nD_r = \begin{vmatrix} {}_nC_1 & {}_nC_2 & {}_nC_3 & {}_nC_4 \cdots & & {}_nC_r \\ 1 & {}_{n-2}C_1 & {}_{n-2}C_2 & {}_{n-2}C_3 \cdots & & {}_{n-2}C_{r-1} \\ 0 & 1 & {}_{n-4}C_1 & {}_{n-4}C_2 \cdots & & {}_{n-4}C_{r-2} \\ 0 & 0 & 1 & {}_{n-6}C_1 \cdots & & {}_{n-6}C_{r-3} \\ \cdots & \cdots & \cdots & \cdots \cdots & & \cdots \\ 0 & 0 & \cdots & 1 & {}_{n-2r+4}C_1 & {}_{n-2r+4}C_2 \\ 0 & 0 & \cdots & & 1 & {}_{n-2r+2}C_1 \end{vmatrix}.$$

The C-symbols are binomial coefficients.

The binomial theorem may be used to obtain powers of x in terms of y_j, $j=1, 2, \cdots, n$, thus:

(4) $x^n = (t + t^{-1})^n = y_n + {}_nC_1 y_{n-2} + {}_nC_2 y_{n-4} + \cdots + {}_nC_{n/2}$, if n is even,

(5) $x^n = (t + t^{-1})^n = y_n + {}_nC_1 y_{n-2} + {}_nC_2 y_{n-4} + \cdots + {}_nC_{(n-1)/2} y_1$, if n is odd.

If n is even the $n/2$ equations (4), $n = 2, 4, \cdots, n$, and if n is odd, the $(n+1)/2$ equations (5), $n = 1, 3, \cdots, n$, may be solved for $y_n = f(x)$. For example,

$$y_6 = x^6 - {}_6C_1 x^4 + \begin{vmatrix} {}_6C_1 & {}_6C_2 \\ 1 & {}_4C_1 \end{vmatrix} x^2 - \begin{vmatrix} {}_6C_1 & {}_6C_2 & {}_6C_3 \\ 1 & {}_4C_1 & {}_4C_2 \\ 0 & 1 & {}_2C_1 \end{vmatrix},$$

$$y_7 = x^7 - {}_7C_1 x^5 + \begin{vmatrix} {}_7C_1 & {}_7C_2 \\ 1 & {}_5C_1 \end{vmatrix} x^3 - \begin{vmatrix} {}_7C_1 & {}_7C_2 & {}_7C_3 \\ 1 & {}_5C_1 & {}_5C_2 \\ 0 & 1 & {}_3C_1 \end{vmatrix} x.$$

THEOREM 1. *The general form for y_n is given in equation* (1), *where the D-coefficients are defined in* (3).

A proof of this theorem by induction will be given. Assume that equation (1) holds for $n = k-2$ and $k-1$. To show that the law is valid for $n = k$, the left member of (1), obtained by setting $n = k-1$, may be multiplied by $t+t^{-1}$ and the right member by x. By transposing and simplifying this resulting equation the proof is completed by demonstrating that the D-coefficients must be connected by precisely the same formula as that which was observed to be the probable law of formation of the above triangle. Namely,

(6) $$ {}_{k-2}D_{s-1} + {}_{k-1}D_s = {}_kD_s.$$

To derive formula (6), the law for constructing the triangle of binomial coefficients,

(7) $$ {}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1},$$

points the way. Hence, in ${}_{k-1}D_s$ add each adjacent column from right to left and apply (7). To this equivalent determinant add ${}_{k-2}D_{s-1}$ written as a determinant of order s, and the result is ${}_kD_s$.

THEOREM 2. *In absolute value* ${}_nD_{n/2} = 2$ and ${}_nD_{(n-1)/2} = n$.

The proof of this theorem follows at once from equation (1) by setting $t = \sqrt{-1} = i$; for then $x = t+t^{-1} = 0$ and $|y_n| = |t^n + t^{-n}| = 2$, if n is even. However, if n is odd, both $t+t^{-1}$ and $t^n + t^{-n} = 0$. In this case, the right member of (1) may be divided by x and the left member by $t+t^{-1}$ and then note that $|\lim_{t \to i}(t^n + t^{-n})/(t+t^{-1})| = n$.

The following four facts are of some interest. If the columns of elements in the above triangle are numbered 0, 1, 2, $\cdots$, k, then beginning with $k = 1$ it is seen that (a) the first term in any column is ${}_{2k}D_k$, (b) the number of terms in any column is $n-(2k-1)$. Furthermore, since the first terms of the successive orders of differences of the terms in any column k are the 2nd term in column $k-1$, the

3rd term in column $k-2$, the 4th term in column $k-3$, $\cdots$ (this follows easily from (6)), then, (c) the jth term, ${}_{2k-1+j}D_k$, in any column may be written as

$$ {}_{2k}D_k + \sum_{u=1}^{k} {}_{j-1}C_u \, {}_{2k-u}D_{k-u}, $$

(d) the sum of j terms may be written as

$$ j \, {}_{2k}D_k + \sum_{u=1}^{k} {}_{j}C_{u+1} \, {}_{2k-u}D_{k-u}. $$

Note: If the parameter t is eliminated from equations (2) by differentiation, a result will be

$$ (4 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0. $$

A solution of this equation in series gives at once the coefficients of equation (1) in the form as shown by Dickson. See the preceding reference.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Logarithms, Trigonometry, Statistics. By H. R. Cooley, P. H. Graham, F. W. John, and A. Tilley. New York, McGraw-Hill Book Company, Inc., 1942. 12+280 pages. $2.00.

As the title of this book indicates, it is a compilation of material for a course in first year college mathematics. However, it contains far more material than can be covered in the usual one semester course for college freshmen, and could be easily used for a two-semester course. Roughly half deals rather briefly with logarithms, including the use of the four principal scales of a simple slide rule, logarithmic and exponential functions, polar coordinates, empirical functions, and statistics.

It is a very well written text book, with clear concise discussions and explanations. It presupposes very little preliminary knowledge of mathematics on the part of the student. On the other hand it is not too elementary for any college class. The reviewer thinks, however, that except for the chapters on trigonometry, which are very complete, it might be improved for classroom use by supplementary exercises.

The chapter on statistics, which covers approximately thirty pages, contains an excellent discussion of classification of data and of central tendencies and dis-

persion. There is a very brief discussion of the normal curve and its uses in statistics, but there is no mention of sampling or of correlation.

BEATRICE L. HAGEN

Differential Equations. By R. P. Agnew. New York, McGraw-Hill Book Company, 1942. 7+341 pages. $3.00.

This text is intended for a "first course" in ordinary differential equations with physical applications. It places unusual emphasis on careful analytical argument and on incentives for the student to reason. Anyone who teaches in this field should find much of interest in the original comments and viewpoints presented frequently with humor, in the book. A poll of a half dozen qualified examiners revealed the unanimous opinion that it would be very desirable to have students who *could* absorb this type of treatment in the usual brief course in elementary differential equations, but that the rather symbolic and critical viewpoint employed would demand a considerable background. We can at least hope that the standards of elementary mathematical education will improve.

As to the content, an "epsilon-delta treatment" of continuity, differentiation, and Riemann and Cauchy integration is presented after a preliminary discussion of some basic definitions. Then first order equations and some physical applications are considered in terms of standard "tricks." (The unifying principle of exact differentials is developed much later in the text.) Division by variables is avoided as a trap for the unwary. The physical basis for the radioactive disintegration equation and the mathematical basis for n-parameter families of curves are considered in an almost annihilatingly critical way which is interesting to those who are aware of silver linings but which is probably shattering to the uninitiated. A factored-operator treatment is given for linear equations of the second and higher orders, supplemented by a chapter on power series in relation to the hypergeometric, the Bessel and the Legendre differential equations. Complex exponentials and impedances and reciprocal operators are used to discuss n-th order ordinary differential equations with constant coefficients, Heaviside operational methods receiving a fairly extensive treatment in terms of general symbols. Application is made to mechanical and electrical problems with special reference to resonance. Unlike most texts in the field, this one does not neglect the important topic of eigenvalues. Some of the more important relevant theorems are stated. There is a chapter on special methods and singular solutions for some first order equations, two chapters on Picard's methods and existence theorems, and finally an appendix on the simple differential equations for slightly bent beams.

The book seems to be remarkably free from misprints, though the following errors may be noted: on page 81 c should be $\sqrt{c}$ in the double inequality; on page 186 (at the bottom) "all $f(x)$" is too inclusive; on page 195, lines 9 and 10 in 9.01, at least one "in" should be dropped. The decimal notation for paragraphs and equations is used and is convenient except when a defenceless little equation is loaded down with a label such as 12.5831 in small type. Numerous informal

problems are provided, the usual "drill" being contrary to the author's tenets.

Frequent discouraging reference to possible misrepresentation of physical phenomena due to the introduction of approximations, seems to the reviewer to be misleading, since nonsingular variation of coefficients can be shown to produce nonsingular variations in solutions. Furthermore, idealizations (approximations guided by experimental knowledge) are the rule rather than the exception in science. However, we can all sympathize with the author's reaction against the usual uncritical attitude of most elementary texts.

J. K. L. MACDONALD

Differential Equations. By Max Morris and O. E. Brown. Revised edition. New York, Prentice-Hall, 1942. 12+355 pages. $3.00.

The first edition of this book was reviewed in this MONTHLY, vol. 41 (1934), pp. 183–4. After five printings the authors and publishers have now issued a revised edition. By comparing it with the first edition, it may be seen that the book is more changed in appearance than in content. A differently-colored cover, a new style of type, and a larger page are examples of alterations in its physical appearance. Changes in the text itself are few. The most extensive are in the chapter on Linear Differential Equations, which has been entirely rewritten in the interests of clarity. New material includes Milne's method of numerical integration and Laplace's equation. A table of integrals and a table of natural logarithms add to the usefulness of the book. Many new problems have been included.

The reviewer noted few misprints, the most serious being the omission of equation (15) from the Existence Theorem on page 8. In equation (19) on page 87, students may be confused by the use of k in two different senses.

The revised edition will appeal, as did the first edition, to those teachers who desire a text in differential equations which emphasizes physical applications and at the same time gives the student an introduction to some advanced topics in algebra and analysis.

H. M. GEHMAN

Spherical Trigonometry. By R. W. Brink. New York, D. Appleton-Century Company, 1942. 8+62 pages. $0.75.

This little book begins with a review of the geometry of the sphere, featuring spherical distances, spherical angles and triangles, and polar triangles. Then comes the trigonometry of a right spherical triangle, including Napier's Rules and the discussion of species. This is followed by a similar discussion of oblique triangles. The law of sines, law of cosines, both for sides and for angles, and the half-angle and half side formulas are fully treated. The discussion of Napier's analogies is followed by a rather full treatment of the ambiguous cases. A short final chapter is devoted to the celestial sphere. Appendices explain the mil, and the haversine. The latitude and longitude of a number of places is included, but the typography is rather crowded, and not clear. Otherwise the type and the

figures are good. The rather brief treatment is not essentially different from that in other texts, hardly justifying the claims of originality in the preface.

VIRGIL SNYDER

A Mathematics Refresher. By A. Hooper. New York, Henry Holt and Co., 1942. 10+342 pages. $1.90.

This is not only an excellent book for candidates for the air forces who have inadequate mathematical background, but it is also valuable for any teacher of mathematics. Mr. Hooper's long and successful experience in teaching air-crew candidates in the R.A.F. and R.C.A.F. has eminently qualified him to write a preparatory text for those who have forgotten their school mathematics, and who must, in the shortest possible time, become completely familiar with the mathematics essential to the aviator. Mr. Hooper knows what to omit and what to emphasize—that is, he understands the difficulties of young people who do not "take to" mathematics—and the result is a minimum, yet thorough foundation, forcefully and enthusiastically presented.

As to the use of *A Mathematics Refresher* in "civilian" classrooms, how many of us have frequently sighed for such a volume! This is the book that we have looked for to knit together arithmetic, algebra, geometry, trigonometry, and the beginnings of the calculus. While we shall regret that it is too brief to use as a class text, we can discover in it innumerable suggestions for our teaching and questions (which, with their answers, have all been tested and approved in aviation schools) for review. And this is the book to hand to John and Mary as a "reference" when they tell us that they have forgotten how to "do" per cent, or that they never did understand what "angle" really means or why anyone ever bothers with logarithms or that mysterious dy/dx!

Mr. Hooper has indeed written "mathematics without tears," but in so doing he has not sacrificed substance. His arrangement is logical, well balanced, and sufficiently detailed for his purpose, and the easy charm of his style adds to the value of the book for all who use it.

MURIEL BOWDEN

NEW BOOKS RECEIVED

A Start in Meteorology. By A. N. Spitz. New York, Norman W. Henley Publishing Co., 1942. 95 pages. $1.50.

Air Navigation for Beginners. By S. G. Lamb. New York, Norman W. Henley Publishing Co., 1942. 103 pages. $1.50.

Learning to Navigate. By P. V. H. Weems and W. C. Eberlee. Second edition. New York, Pitman Publishing Corp., 1943. 8+135 pages. $2.00.

A Handbook of Perspective Drawing. By J. C. Morehead and J. C. Morehead, Jr. Pittsburgh, James C. Morehead, 1941. 6+166 pages. $4.50.

Empirical Equations and Nomography. By D. S. Gale. New York and London, McGraw-Hill Book Co., Inc., 1943. 9+200 pages. $2.50.

Differential Equations. By H. W. Reddick. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1943. 9+245 pages. $2.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 571. *Proposed by Sol Mitchell, University of Toronto*

Let O be the midpoint of a chord AB of a circle, and let CD, EF be any two other chords through O. Prove synthetically that CE and DF meet AB in points equidistant from O.

E 572. *Proposed by V. Thébault, San Sebastián, Spain*

What scale of notation admits perfect squares ab, bb, and cd, where a, b, c, d are consecutive integers?

E 573. *Proposed by N. A. Court, University of Oklahoma*

Given two (three) vertices of a triangle (tetrahedron), determine the remaining vertex so that a given point and a given line (plane) shall be harmonic for the triangle (tetrahedron).

E 574. *Proposed by W. E. Buker, Pittsburgh Public Schools*

If a quadrilateral with sides a, b, c, x is inscribed in a semicircle of diameter x, show that

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$

E 575. *Proposed by P. S. Donchian, Hartford, Connecticut*

Let (n^r) denote the standard residue of n^r modulo p (an odd prime), so that $(n^r) \equiv n^r \pmod{p}$ and $0 \leq (n^r) < p$. Prove that the $p-2$ numbers

$$\sum_{n=1}^{p-1} (n^r) \qquad (r = 1, \cdots, p-2)$$

are a permutation of the $p-2$ numbers

$$\sum_{r=0}^{p-2} (n^r) \qquad (n = 2, \cdots, p-1),$$

and that all these numbers are divisible by p.

SOLUTIONS

Stirling's Numbers of the First Kind

E 520 [1942, 257]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Given

$$u_n = n!\sum_1^n \frac{1}{r}, \quad \text{prove that} \quad \lim_{r\to\infty} \frac{u_{r+1}}{\Delta^r u_1} = e.$$

Solution by H. W. Becker, Mare Island Submarine Base. In the notation of Jordan (*Tohoku Mathematical Journal*, vol. 37, 1933, pp. 255–283), Stirling's numbers of the first kind are given by

$$S_{n+1}^m = S_n^{m-1} - nS_n^m, \qquad S_n^0 = 0 \ (n > 0), \qquad S_n^n = 1.$$

On comparing this recurrence formula with the proposer's

$$u_n = (n-1)! + nu_{n-1}, \qquad u_1 = 1,$$

we see that $u_n = |S_{n+1}^2|$. Thus the desired result can be obtained by setting $m=2$ and $n=1$ in the more general relation

$$\lim_{r\to\infty} \frac{\Delta^r |S_{n+1}^m|}{|S_{r+n+1}^m|} = \frac{1}{e},$$

which we proceed to prove.

From Jordan's asymptotic expression

$$|S_{n+1}^m / S_n^m| \sim n \tag{1}$$

(*loc. cit.*, p. 261), we deduce

$$|S_{n+1}^m / S_{n+1-j}^m| \sim (n)_j = n(n-1)\cdots(n+1-j),$$

whence (for large r)

$$\begin{aligned} \frac{\Delta^r |S_{n+1}^m|}{|S_{r+n+1}^m|} &= \frac{\sum_{j=0}^r (-1)^j \binom{r}{j} |S_{r+n+1-j}^m|}{|S_{r+n+1}^m|} \sim \sum_{j=0}^r \frac{(-1)^j \binom{r}{j}}{(r+n)_j} \\ &= \sum_{j=0}^r \frac{(-1)^j}{j!} \frac{(r)_j}{(r+n)_j} \sim \sum_{j=0}^r \frac{(-1)^j}{j!} \left(\frac{r}{r+n}\right)^j \\ &\sim e^{-r/(r+n)}. \end{aligned} \tag{2}$$

Thus the limit, as r tends to ∞, is e^{-1}, as required.

Since $|S_{n+1}^1| = n!$, another special case of this asymptotic formula is

$$\frac{\Delta^r n!}{(r+n)!} \sim e^{-r/(r+n)} \qquad (r \to \infty).$$

Instead of keeping n fixed, we may let it be a constant multiple of r; *e.g.*

$$\lim_{r\to\infty} \frac{\Delta^r r!}{(2r)!} = e^{-1/2} = .6065\ldots. \tag{3}$$

The convergence is quite rapid; when $r=3$ we already have

$$\frac{\Delta^3 3!}{6!} = \frac{426}{720} = .5916\dot{6}.$$

Editorial Note. The approximation (2) can be justified as follows. Stirling's formula $n!\sim(2\pi)^{1/2}n^{n+1/2}e^{-n}$ gives

$$\begin{aligned}\frac{(r)_j}{(r+n)_j} &= \frac{r!(r+n-j)!}{(r+n)!(r-j)!}\\ &\sim \left(\frac{r}{r+n}\right)^j \left(1-\frac{j}{r}\right)^{-r+j-1/2}\left(1-\frac{j}{r+n}\right)^{r+n-j+1/2}\\ &\sim \left(\frac{r}{r+n}\right)^j \left(\frac{1-j/r}{1-j/(r+n)}\right)^{j-1/2} \sim \left(\frac{r}{r+n}\right)^j \left(1-\frac{nj(j-\frac{1}{2})}{r(r+n)}\right).\end{aligned}$$

Hence the error in (2) is approximately

$$\begin{aligned}&-\sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{r}{r+n}\right)^j \frac{nj(j-1+\frac{1}{2})}{r(r+n)}\\ &= -\sum_{j=2}^{\infty}\frac{1}{(j-2)!}\left(-\frac{r}{r+n}\right)^{j-2}\frac{rn}{(r+n)^3}\\ &\quad+\sum_{j=1}^{\infty}\frac{1}{(j-1)!}\left(-\frac{r}{r+n}\right)^{j-1}\frac{n}{2(r+n)^2}\\ &= -e^{-r/(r+n)}\left(\frac{rn}{(r+n)^3}-\frac{n}{2(r+n)^2}\right) = -e^{-r/(r+n)}\frac{n(r-n)}{2(r+n)^3}.\end{aligned}$$

In the special case when $m=1$, (1) is exact, so we can say confidently that, for large r,

$$\frac{\Delta^r n!}{(r+n)!}\sim e^{-r/(r+n)}\left(1-\frac{n(r-n)}{2(r+n)^3}\right).$$

The presence of the factor $r-n$ explains the rapid convergence of (3).

Eight Consecutive Digits forming a Square

E 538 [1942, 546]. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect square whose digits form one of the permutations of eight consecutive digits. (Cf. E 532.)

Solution by E. P. Starke, Rutgers University. We consider three possible cases:

I. Let the digits of N^2 be 0, 1, 2, 3, 4, 5, 6, 7, in some order. Then

(1) $$N^2 \equiv 1, \qquad N \equiv \pm 1 \pmod 9.$$

Also, from consideration of the final two digits of N^2, we have

(2) $$\pm N \equiv 1, 2, 4, 5, 6, 8, 11, 15, 16, 18, 19, 21, 24, \text{or } 25 \pmod{50}.$$

Furthermore, if we do not permit an initial zero, we have

(3) $$3199 < N < 8749.$$

Now put $N=100a+b$, $0<b<100$. To each value of b which is satisfactory according to (2), there correspond a number of values of a which satisfy (1) and (3). The possibilities for N are thus restricted to something over six hundred. Testing these (or referring to a table of squares, *e.g.* Barlow's) discloses six squares of the proposed form:

$$3698^2 = 13675204, \quad 4175^2 = 17430625, \quad 4616^2 = 21307456,$$
$$5968^2 = 35617024, \quad 6596^2 = 43507216, \quad 7532^2 = 56731024.$$

II. Let the digits of N^2 be 1, 2, 3, 4, 5, 6, 7, 8. Then

$$N \equiv 0 \pmod 3, \qquad 3513 < N < 9363,$$
$$\pm N \equiv 4, 5, 6, 8, 9, 11, 15, 16, 18, 19, 21, 22, 24, 25 \pmod{50}.$$

Proceeding as in Case I, we find five squares:

$$3678^2 = 13527684, \quad 5904^2 = 34857216, \quad 8082^2 = 65318724,$$
$$8559^2 = 73256481, \quad 9024^2 = 81432576.$$

III. Let the digits of N^2 be 2, 3, 4, 5, 6, 7, 8, 9. Then

$$N^2 \equiv 8 \pmod 9.$$

Since 8 is not a quadratic residue modulo 9, this last case produces no results.

Also solved by W. E. Buker.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4082. *Proposed by H. S. M. Coxeter, University of Toronto*

In hyperbolic geometry, obtain the condition

$$\cos \tfrac{1}{2}A < \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C$$

for the triangle ABC to have a proper excircle beyond BC.

4083. *Proposed by P. Erdös, Princeton, N. J.*

Let $a_1 < a_2 < \cdots < a_x \leqq n$ be an arbitrary sequence of positive integers such that no a_i divides the product of the others, then $x \leqq \pi(n)$, where $\pi(n)$ denotes the number of primes not exceeding n.

4084. *Proposed by Otto Dunkel, Washington University*

On the sides A_iA_k of a given triangle $A = A_1A_2A_3$ as bases, directly similar triangles $B_iA_iA_k$ are constructed interiorly giving the triangle $B = B_1B_2B_3$. Show that, if B has the maximum area when the sense of rotation of its vertices is opposite to that for A, the triangles $B_iA_iA_k$ must be isosceles with $\cot \alpha \cot V = 3$, where α is the base angle and V is the Brocard angle for A. Determine the form of the triangle B giving the maximum.

4085. *Proposed by V. Thébault, San Sebastián, Spain*

Given an equilateral hyperbola (H) and a circle (O) passing through the center ω of (H), show that the necessary and sufficient condition for the existence of an infinite number of triangles inscribed in (H) and circumscribing the circle is that the center O of the circle lies on (H). Consider the envelope of the sides of these triangles.

SOLUTIONS

Tetrahedron, Hyperbolic Set of Lines

3986 [1941, 152]. *Correction. Proposed by V. Thébault, San Sebastián, Spain*

The six points $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are taken respectively on the edges BC, CA, AB, DA, DB, DC of the tetrahedron $ABCD$, and the radical planes of the circumsphere of $ABCD$ with the spheres $(A\beta\gamma\alpha')$, $(B\gamma\alpha\beta')$, $(C\alpha\beta\gamma')$, $(D\alpha'\beta'\gamma')$ meet the planes of the faces BCD, CDA, DAB, ABC in the straight lines Δ_1, Δ_2, Δ_3, Δ_4. Show that these four straight lines are generators of the same system of a quadric surface if the straight lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent, and conversely.

Generalization by R. Bouvaist, Nantes, France

THEOREM I. *The circles (ω_1), (ω_2), (ω_3) pass respectively through the vertices A, B, C of the triangle ABC whose circumcircle is (O). The radical axis of (ω_2) and (ω_3) meets BC in the point α; the radical axis of (ω_3) and (ω_1) meets CA in β; the radical axis of (ω_1) and (ω_2) meets AB in γ. The radical axis of (O) and (ω_1) meets BC in α'; the radical axis of (O) and (ω_2) meets CA in β'; the radical axis of (O) and (ω_3) meets AB in γ'. If the straight lines $A\alpha$, $B\beta$, $C\gamma$ are concurrent, the points α', β', γ' are collinear, and conversely.*

If $S=0$ is the equation of (O) and $\Delta=0$ that of the line at infinity, we have the following equations

$$(1)\qquad \begin{gathered}(\omega_1):\ S+\Delta(m_1y+n_1z)=0;\qquad (\omega_2):\ S+\Delta(l_2x+n_2z)=0;\\ (\omega_3):\ S+\Delta(l_3x+m_3y)=0.\end{gathered}$$

The radical axes of (ω_2) and (ω_3), of (ω_3) and (ω_1), of (ω_1) and (ω_2) have the equations

$$(2)\qquad \begin{gathered}(l_2-l_3)x-m_3y+n_2z=0;\qquad l_3x+(m_3-m_1)y+n_1z=0;\\ -l_2x+m_1y+(n_1-n_2)z=0.\end{gathered}$$

If $A\alpha$, $B\beta$, $C\gamma$ are concurrent, we find that $m_1n_2l_3=n_1l_2m_3$. If the straight lines whose equations are

$$(3)\qquad m_1y+n_1z=0;\qquad l_2x+n_2z=0;\qquad l_3x+m_3y=0$$

meet BC, CA, AB in the collinear points α', β', γ', we have the same relation $m_1n_2l_3=n_1l_2m_3$.

THEOREM II. *The four spheres* (ω_1), (ω_2), (ω_3), (ω_4) *pass respectively through the vertices* A, B, C, D *of the tetrahedron* $ABCD$ *whose circumsphere is* (O). *The radical planes of* (ω_1) *and* (ω_2), *of* (ω_3) *and* (ω_4), *of* (ω_1) *and* (ω_3), *of* (ω_2) *and* (ω_4), *of* (ω_1) *and* (ω_4), *of* (ω_2) *and* (ω_3) *meet respectively* AB, CD, AC, DB, DA, BC *in the points* $\gamma, \gamma', \beta, \beta', \alpha', \alpha$. *The radical planes of sphere* (O) *with each of the spheres* (ω_1), (ω_2), (ω_3), (ω_4) *meet respectively the planes of the faces* BCD, CDA, DAB, ABC *in the straight lines* Δ_1, Δ_2, Δ_3, Δ_4. *If the straight lines* $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ *are concurrent, then* Δ_1, Δ_2, Δ_3, Δ_4 *are on the same hyperboloid, and conversely.*

The radical plane of (ω_1) and (ω_2) has the equation

$$(4)\qquad -l_2x+m_1y+(n_1-n_2)z+(p_1-p_2)t=0$$

and the equations of the five other radical planes are written in a similar manner. If the point $P(x_0, y_0, z_0, t_0)$ is the point of concurrency of $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, we find from the equations (4) that $l_2x_0=m_1y_0$, and in the same way five other equations. We write these in the form

$$(5)\qquad \begin{gathered}l_2=m_1y_0/x_0;\qquad l_3=n_1z_0/x_0;\qquad l_4=p_1t_0/x_0;\qquad n_2=m_3y_0/z_0;\\ p_2=m_4y_0/t_0;\qquad p_3=n_4z_0/t_0.\end{gathered}$$

The equations of the radical planes of (O) and (ω_i) may then be written

$$(6)\qquad \begin{gathered}m_1y+n_1z+p_1t=0;\qquad m_1x/x_0+n_2z/y_0+p_2t/y_0=0;\\ n_1x/x_0+n_2y/y_0+p_3t/z_0=0;\qquad p_1x/x_0+p_2y/y_0+p_3z/z_0=0;\end{gathered}$$

and we see that they are polar planes of A, B, C, D with respect to the quadric surface

$$(7)\qquad m_1xy/x_0+n_1xz/x_0+p_1xt/x_0+n_2yz/y_0+p_2yt/y_0+p_3zt/z_0=0.$$

The straight lines Δ_i are therefore on the same hyperboloid. Conversely, if they are on the same hyperboloid, we have the relations

$$(8) \qquad \begin{gathered} n_2p_3m_4 = p_2m_3n_4; \qquad n_1p_3l_4 = p_1l_3n_4; \qquad m_1p_2l_4 = p_1l_2m_4; \\ m_3l_2n_1 = m_1l_3n_2. \end{gathered}$$

The first, for example, means that Δ_2, Δ_3, Δ_4 meet CD, DB, BC in three points which are collinear. The relations above are seen to be satisfied by means of the equations (5); and this proves the proposition.

Note by the Proposer. The second theorem results immediately from the first by observing that the planes of the faces cut circles on the spheres (ω_i), and that the straight lines Δ_i meet each of four straight lines $d_i \equiv (\alpha_i', \beta_i', \gamma_i')$ which correspond to them in the faces of the tetrahedron.

Editorial Note. The proposer stated in regard to the original problem that its theorem is easily deduced from a theorem by Roy in the *Nouvelles Annales de Mathématiques*, 1926, p. 277, which is the special case of Theorem I where the circles (ω_i) intersect in pairs on the sides of ABC. For Theorem II there are a number of special cases which we shall separate by assuming that no sphere (ω_i) passes through two vertices of $ABCD$; then the point $P(x_0, y_0, z_0, t_0)$ in the above proof does not lie in a face and no line Δ_i passes through a vertex. As the proposer's note to the above solution states, it is easy to show synthetically from Theorem I that, if $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ meet in a point P, then there exist four straight lines d_i each lying in a corresponding face which meet each of the four lines Δ_i. The synthetic proof of the converse is long and complicated with exceptional cases, and even when we make the above assumption it is not simple. In the case now considered one method is to show that the straight lines AP_1, BP_2, CP_3, DP_4, defined below, meet in a point P and then that $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent in this point. It is simpler to prove together the two theorems analytically as follows. If in the two sets of equations for the radical axes we set $t=0$, we find from (4) and similar equations of the solution that the equations of the traces $A\alpha$, $B\beta$, $C\gamma$ in the plane of ABC of the planes $A\alpha'\alpha$, $B\beta'\beta$, $C\gamma'\gamma$ meeting in the point P are

$$(1) \qquad m_3y - n_2z = 0; \qquad l_3x - n_1z = 0; \qquad l_2x - m_1y = 0.$$

The necessary and sufficient condition that they meet in the point P_4 which is the trace of DP in the face ABC is $l_2m_3n_1 - l_3m_1n_2 = 0$. The traces of Δ_1 on BC, of Δ_2 on CA, of Δ_3 on AB are the points

$$(2) \qquad \alpha_1\colon\ 0, n_1, -m_1, 0; \qquad \beta_2\colon\ n_2, 0, -l_2, 0; \qquad \gamma_3\colon\ m_3, -l_3, 0, 0.$$

The necessary and sufficient condition that α_1, β_2, γ_3 lie on a straight line d_4 is the same as above. By cyclic permutations of the letters and subscripts we then find for the existence of d_1, d_2, d_3 the necessary and sufficient conditions in the first three equations of (8) in the above solution, the last being for d_4. These four equations are not independent. Conversely, if there exist four such lines d_i, then we have shown that for d_4 the equations (1) are consistent; and by cyclic permu-

tation we have those for d_1, d_2, d_3 which are also consistent. We then have twelve equations with duplication of each; the six distinct equations are in pairs the equations of $\alpha\alpha', \beta\beta', \gamma\gamma'$ and as pointed out they are consistent by reason of the equations (8). Hence these six equations have a unique solution (x_0, y_0, z_0, t_0). and we have now proved that $\alpha\alpha', \beta\beta', \gamma\gamma'$ meet in a point P with these coordinates. Thus the necessary and sufficient conditions that $\alpha\alpha', \beta\beta', \gamma\gamma'$ meet in a point are the same as those for the existence of the straight lines d_i. The four lines d_i are distinct; for, if two coincide they coincide with an edge of $ABCD$ and then two of the lines Δ_i pass respectively through the vertices on that edge, and this is at present excluded.

But in order for the four lines Δ_i to form a hyperbolic set we must impose the condition that no two lie in a plane; and we have then to consider the two special cases where they do not form a hyperbolic set. First there is the case where only two determine the same plane, say Δ_1 and Δ_2. It will then be seen synthetically that $d_1, d_2, \Delta_1, \Delta_2$ are concurrent in γ_{34}' on CD, the planes (Δ_1, Δ_2), (d_1, d_2) meet face ABC in d_4, Δ_4 respectively. It then follows that $d_3, d_4, \Delta_3, \Delta_4$ are concurrent in γ_{12} on AB, the point γ_{34}' lies in the plane (Δ_3, Δ_4) and γ_{12} in the plane (Δ_1, Δ_2), that is the planes (Δ_1, Δ_2) and (Δ_3, Δ_4) intersect in the straight line through γ_{12} and γ_{34}'. In this case there are an infinite number of straight lines in the plane (Δ_1, Δ_2) that meet Δ_3 and Δ_4, in γ_{12}, and similarly for plane (Δ_3, Δ_4). We now have the case where, for example, Δ_1 and Δ_2 lie in a plane and also Δ_1 and Δ_3 lie in a plane. It then follows from the above that $d_1 = \Delta_1$, $d_3 = \Delta_3$, $d_2 = \Delta_2$, and finally $d_4 = \Delta_4$. Thus the four lines Δ_i lie in the same plane, and we now have the condition lacking in the original statement of the problem. The remaining special cases relate to the number of distinct lines d_i. In the case where there are precisely three distinct lines d_i and no two of the lines Δ_i lie in a plane, the four lines Δ_i form a hyperbolic set. The discussion of the remaining cases is tedious, and it is simpler to assume that the theorem excludes them.

Properties of a Special Triangle

4026 [1942, 128]. *Proposed by V. Thébault, San Sebastián, Spain*

(1) Construct a triangle ABC knowing a, A and given that the median and symmedian from A are perpendicular and parallel to two given directions. (2) Indicate the properties of this special triangle. (3) Let B' and C' be the projections of B and C on a variable straight line AP which cuts BC in P. The locus of the harmonic conjugate of P with respect to B' and C' is a right strophoid having the vertex A for a double point and tangent to the bisectors of angle A.

Editorial Note. The proposer stated: (1) The given elements $a = BC$ and the angle A suffice to determine the circumradius R of ABC. Let A_m be the midpoint of BC; let M be the point of intersection of BC with the symmedian from A whose direction is known; and let F be the other point of intersection of AM and the circumcircle (O). The bisectors of angle A and A_mAM intersect (O) again at the midpoints of the arcs BC, and the median AA_m meets (O) again at the point E diametrically opposite to F, since A_mAM is a right angle. The points E, F be-

ing known as also the directions of AA_m, AF, the vertex A is determined from the points E, F.

(2) This special triangle is rather curious. The circle with A_mM as diameter is orthogonal to the Apollonian circle for BC. We have the relations

$$a^2 = -2bc\sqrt[3]{\cos A}\,(\sqrt[3]{\cos A} + 1), \qquad A > 90°,$$
$$(b^2 + c^2)/2bc = -\sqrt[3]{\cos A} = m_a/s_a,$$

where m_a and s_a denote the lengths of AA_m and AM.

(3) Consider the parabola tangent to BC and the interior bisector AI, and having AA_m for directrix. The locus of the harmonic conjugate of P with respect to B', C' is the pedal of the parabola with respect to the point A, and it is therefore a right strophoid. The double point is A, and the bisectors of angle A are the tangents at that point. The curve passes through the isogonal centers of ABC.

Boolean Algebras

4027 [1942, 202]. *Proposed by Morgan Ward, California Institute of Technology*

In the projective plane P, a triangle with sides A, B, C and vertices D, E, F is the only linear configuration forming a Boolean algebra of order eight with respect to the operations of union and cross-cut. Here the union of A and B is P, the cross-cut of E and D is the null space Z, and so on. In a three dimensional space P, two types of Boolean algebra of order eight are possible; (i) the configuration of three planes A, B, C through a point Z meeting in three lines D, E, F; and (ii) the configuration of two planes A and B through a line D and a line C skew to D meeting A and B in points E and F. Here Z again is the null space.

Show that in a projective space of n dimensions, the total number of distinct types of linear configuration forming a Boolean algebra of order eight is asymptotically equal to $n^3/3!3!$. Also show that the corresponding number for a Boolean algebra of order 2^r is asymptotically equal to $n^r/r!r!$.

Solution by the Proposer. The actual number $u(n)$ of types of linear configurations in a projective space of $n-1$ dimensions forming a Boolean algebra of order eight is given by the formula

$$u(n) = \left[\frac{2n^3 + 3n^2 - 6n}{72}\right] \qquad \text{if} \quad n \equiv 0, 2, 4, 5 \pmod 6$$
$$= \left[\frac{2n^3 + 3n^2 - 6n}{72}\right] + 1 \quad \text{if} \quad n \equiv 1, 3 \pmod 6.$$

For small n this gives the table

Dimensions:	1	2	3	4	5
Number:	0	1	2	4	7

which it is easy to verify directly.

The asymptotic formula follows immediately but one can establish it for a Boolean algebra of order 2^r by a simple argument which I sketch here. One simply observes that the Boolean algebra as a configuration is completely determined by the dimensions $\pi_1, \cdots, \pi_r$ of the linear manifolds $P_1, \cdots, P_r$ which constitute the "points" of the lattice algebra. If the space is of n dimensions, then

$$\pi_1 + \pi_2 + \cdots + \pi_r = n + \epsilon \tag{1}$$

where

$$0 \leqq \epsilon \leqq n - r; \qquad \pi_i \geqq \epsilon + 1, \tag{2}$$

and ϵ is the dimension of the null element of the Boolean algebra. (For simplicity of notation I count a point as of dimension one, a line of dimension two, and so on, so that the "dimension" here is one more than the ordinary dimension I used in stating the problem.)

Since we are interested in distinct types, we add the restriction

$$\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_r. \tag{3}$$

Let $u(n)$ be total number of solutions of the diophantine system (1), (2), (3). Then

$$u(n) - u(n-1) = \phi(n)$$

where $\phi(n)$ is the total number of solutions of

$$\pi_1 + \pi_2 + \cdots + \pi_r = n \tag{4}$$

$$\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_r \geqq 1. \tag{5}$$

Now the total number of solutions of (4) with (5) replaced by

$$\pi_i \geqq 1 \tag{6}$$

is evidently

$$(n-1)(n-2)\cdots(n-r+1)/(r-1)!.$$

But if $n>1$, solutions of (4) and (6) with all π unequal outnumber all others. Hence we infer $\phi(n)$ is asymptotic to

$$\frac{(n-1)(n-2)\cdots(n-r+1)}{r!(r-1)!}.$$

Hence $u(n)$ is asymptotic to

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!r!} \quad \text{or} \quad u(n) \sim \frac{n^r}{r!r!}.$$

The details of a rigorous proof are easily provided since we are dealing with *polynomials* in n essentially.

Geodesics

4033 [1942, 263]. *Proposed by P. D. Thomas, Norman, Oklahoma*

Points on a surface with the linear element $ds^2=[(u+a)^2+(v+b)^2](du^2+dv^2)$ correspond to points in the xy-plane by $x=u$, $y=v$. Show that geodesics on the surface correspond to equilateral hyperbolas in the xy-plane.

Solution by Thomas Bauserman, Camp Davis, N. C. By making the transformation $(u+a)=\rho\cos\theta$, $(v+b)=\rho\sin\theta$, the linear element becomes

$$ds^2 = \rho^2(d\rho^2 + \rho^2 d\theta^2).$$

Then by the transformation, $\rho\, d\rho = dr$, and $2\theta=\phi$

$$ds^2 = dr^2 + r^2 d\phi^2.$$

Since this is the polar form of the linear element of a straight line, the equation of the geodesics may be written

$$2Ar\cos\phi + 2Br\sin\phi = C,$$

which becomes

$$A\rho^2\cos 2\theta + B\rho^2\sin 2\theta = C,$$

or

$$\rho^2\cos 2(\theta-\alpha) = K,$$

and finally

$$\rho^2\cos^2(\theta-\alpha) - \rho^2\sin^2(\theta-\alpha) = K.$$

In the xy-plane this is an equilateral hyperbola where the origin has been translated to the point $(-a, -b)$ and the axes rotated through an angle α.

A set of geodesics is also given by $\phi=$constant which are represented in the xy plane by straight lines through $(-a, -b)$.

Solved also by the proposer using the differential parameter $\Delta_1 f$ (Eisenhart's *Differential Geometry*, 1909, p. 217).

FIFTH ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The fifth annual meeting of the Northern California Section of the Mathematical Association was held at the University of San Francisco on Saturday, January 30, 1943. Professor E. B. Roessler, vice-chairman of the Section, presided at both morning and afternoon sessions in the absence of Dean Fredrick Wood, the chairman, who was unable to be present. During the noon recess, members and visitors were the guests of the University of San Francisco at a luncheon served in "The Lounge" of the University.

The attendance at the two sessions was seventy-five, including the following twenty-four members of the Association: H. M. Bacon, G. A. Baker, G. C.

Evans, E. L. Fitzgerald, S. A. Francis, Emma V. Hesse, D. H. Lehmer, Sophia H. Levy, A. L. McCarty, E. D. Miller, F. R. Morris, E. J. Moulton, E. J. Phillips, George Polya, Edris P. Rahn, E. B. Roessler, Ethel Spearman, Gabor Szegö, Ruth G. Sumner, K. J. Waider, R. K. Wakerling, L. A. Walker, Harriet A. Welch, A. R. Williams.

The following officers were elected for the coming year: Chairman, E. B. Roessler, University of California at Davis; Vice-Chairman, Gabor Szegö, Stanford University; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was re-elected to represent the Section as associate editor of the *California Journal of Secondary Education*. A resolution of thanks to the University of San Francisco for its generous hospitality was unanimously adopted for transmission to the Very Reverend William J. Dunne, President of the University. The report of Professor Levy regarding the activities of the Committee on Mathematical Education of the Northern and Southern California Sections was followed by the adoption of a resolution endorsing the following statement recently adopted by the Subcommittee on Mathematics of the California Committee for the Study of Education:

It is the opinion of the Subcommittee on Mathematics that our California schools and their teachers can make an essential and invaluable contribution to the war effort by continuing to teach all of the mathematics which has formed and now forms a most important part of their work. Students are leaving school at every grade and age to enter either industry or the armed forces in capacities where they find mathematics a prime requirement.

A fourth-year high school student without skill in arithmetic should, at the time, be given arithmetic intensively, followed by as much algebra as can be included. A third-year high school student without suitable skill in arithmetic should be started with intensive training in this subject. For him there will be time, also, for algebra through quadratics and some geometry. It will be noted that these students are more mature than the usual beginner and also know their immediate need for mathematical training when they start these basic subjects. Hence, they can cover the materials at a relatively faster pace.

For students in the earlier grades, there will be time for the usual sequence of algebra, geometry, and trigonometry. For these students the question of time is not pressing, and the mathematics courses already being given in many schools are suitable.

This Subcommittee agrees with the opinions of many representatives of the Armed Services and industry that the mathematics courses in schools must continue to place emphasis on basic and fundamental parts of the subject treated. Such applications as are given, and there should be as many as possible, should be primarily for the purpose of illustrating and giving training in the basic mathematics. The specific uses of mathematics in the various branches of the Service and industry are best given by appropriate instructors after entrance into service. These are technical subjects and are therefore adequately taught only by the technicians themselves.

When courses are given in basic principles of mathematics, it is clear that such courses fit the needs of both boys and girls, whether for service in the Armed Forces or in industry.

There is general agreement on the point that few students can use arithmetic satisfactorily. It is therefore urged that all teachers beginning with those in our primary schools contribute their utmost towards the training of pupils in the fundamentals of arithmetic. Skills attained at any level should be maintained by cumulative review and use at later levels. Teachers of algebra and geometry should utilize the many opportunities which exist in their courses for continued use of arithmetic. Also, the teachers of geometry should further develop algebraic skill through having their students use it in geometry.

It is the unanimous opinion of this Subcommittee on Mathematics that our schools will not be

rendering their greatest possible service to the Army, Navy, and war industries, unless they conscientiously urge their students to take basic mathematics, including arithmetic, algebra at least through quadratics, concepts and constructions of plane and solid geometry, and trigonometry.

Mrs. Sumner reported that the *California Journal of Secondary Education* would publish a symposium of articles on mathematics in the near future, probably in the May issue.

By invitation of the Section, Professor J. H. McDonald of the University of California gave an hour's address during the morning session.

The following papers were read:

1. "Analogy, a course of discovery" by Professor George Polya, Stanford University.

2. "The mathematical requirements of a course in college physics" by Professor K. J. Waider, University of San Francisco.

3. "Conformal mapping" by Professor J. H. McDonald, University of California, introduced by Professor Levy.

4. "Mathematics in our schools and its contribution to war" (second paper) by Professor Sophia H. Levy, University of California.

5. "The difference-equation method in heat conduction problems" by Professor W. P. Berggren, University of California at Davis, introduced by Professor Roessler.

6. "A solution of $x^2+y^2+z^2=w^2$" by A. L. McCarty, San Francisco Junior College.

7. "Elementary derivation of series for sin x and cos x" by Professor J. V. Uspensky, Stanford University, introduced by the Secretary.

8. "The new military training programs in colleges and universities" by Professor F. R. Morris, Fresno State College.

Abstracts of these papers follow:

1. Professor Polya discussed a well-known but not too commonplace question of elementary solid geometry in order to illustrate the typical way of using analogy in the solution of problems. If the proposed problem does not seem to be easily accessible, we may look out for some more accessible analogous problem. If we succeed in finding and solving such a simpler analogous problem, we may use the *method* of its solution, or we may use its *result* for our original problem, or we may use *both the method and the result* as it happens in the example discussed. The aim of the speaker was to show that methods of discovery are not at all above the level of the classroom, provided the teacher has some real interest and knowledge, and does not spend all his time on technicalities. The subject was taken from a chapter of the book on the solution of problems which the speaker is preparing.

2. Professor Waider stated that of the two lower division courses in general physics commonly offered, the one taken by premedical and letters students requires nothing beyond high school mathematics; while the other, intended for science and engineering majors, involves the use of some calculus. A thorough grounding in the fundamentals of algebra, geometry, and trigonometry con-

stitutes the basic preparation for successful work in a college physics course. Factors of special importance because of their continued employment by the student are: firstly, facility in carrying out numerical computations with accuracy and dispatch, a feat which can scarcely be accomplished without the aid of a slide rule; and secondly, familiarity with graphical representation, for the purpose of depicting the effect of physical laws.

3. Professor McDonald gave in outline three methods of determining the map function for simply connected surfaces. Two of these do not lead to an effective construction of that function, and the third, while suggesting a convergent sequence, proves to be without utility for arithmetic calculation. A fourth method was proposed and illustrated in a manner which shows it to have considerable availability in that regard.

4. Professor Levy stated that during the year many schools have increased the amount of work given in mathematics. The armed services and industry have asked for the teaching of fundamentals, and statements are heard on all sides that fundamentals are now being taught. Inquiries to two-hundred fifty schools have confirmed the belief that the meaning of *fundamentals* has undergone drastic change. Apparently what is fundamental for a navigator is different from what is fundamental for a bombardier! Trigonometry on land and sea is different from trigonometry in the air . . . everything must be "preflight" to be important! It is one thing to do patchwork for the senior in high school who does not know his mathematics, for he needs it quickly. It is another thing to do patchwork for the student who has time in which he could be given real fundamentals. The first duty of our schools is to undertake thorough programs. There is great danger that the impression of being extremely patriotic and extremely busy on patchwork may lead people to ignore the essential job, and on this essential job there is no waste. When the war ends, the fundamental training in mathematics which the boys and girls will have had will be of far more benefit than anything else upon which their time and effort could have been spent.

5. Professor Berggren described a method of solving transient problems to which the Poisson-Fourier (parabolic) partial differential equation is applicable. The procedure, conveniently carried out graphically, is based on repetitive solution of difference equations. The most useful feature of the method is its great versatility with regard to the application of boundary conditions.

6. Mr. McCarty showed that $x^2+y^2+z^2=w^2$ may be written in the form $(w+x)(w-x)=(y+iz)(y-iz)$ or

$$\frac{w+x}{y-iz}=\frac{v+iz}{w-x}=u+iv.$$

This latter form leads to four equations which yield

$$x=ku^2+kv^2-k, \qquad y=2ku, \qquad z=2kv,$$

and

$$w=ku^2+kv^2+k.$$

7. Professor Uspensky gave an elementary derivation of inequalities equivalent to sine and cosine series which does not require even the notion of the limit.

8. Professor Morris stated that in December of 1942 the Army, Navy, and War Manpower Commission issued a joint statement giving preliminary plans for placing the enlisted reserves on active duty and utilizing colleges and universities for giving these men further training of a collegiate grade. The two branches of this plan are known as the Army Specialized Training Program and the Navy Collegiate Training Program. Details of these programs had not been completed, but the membership of advisory committees and executive officers indicate that college faculties and laboratories will be effectively used in the war effort.

H. M. BACON, *Secretary-Treasurer*

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The Society for the Promotion of Engineering Education will hold its Fiftieth Anniversary meeting at the Drake Hotel in Chicago on Friday, Saturday and Sunday, June 18–20, 1943. The following men will speak on the program of the Mathematics Division: Professors W. L. Ayres, G. A. Bliss, J. M. Dobbie, L. R. Ford, A. N. Milgram, M. A. Sadowsky, Evan Thomas, J. H. Zant. Professor L. R. Ford, Illinois Institute of Technology, is Chairman and Professor J. W. Cell, North Carolina State College, is Secretary of the Mathematics Division.

Professor J. M. Howie of Nebraska Wesleyan University, Lincoln, Nebraska, wishes to dispose of a complete set (49 volumes) of the AMERICAN MATHEMATICAL MONTHLY, bound and in good condition.

Dr. P. H. Anderson of John Carroll University is also serving as a statistician with the regional office of the War Production Board.

Dr. Henry Barone of The Citadel has been appointed assistant professor at Pennsylvania State College.

Associate Professor F. C. Gentry, on leave from Louisiana Polytechnic Institute, is at present a lecturer in the pre-meteorological program of the University of New Mexico.

Dr. J. F. Heyda of Michigan State College has been appointed assistant professor at Denison University.

Assistant Professor Tryphena Howard of Western Kentucky State Teachers College has resigned. She was married in December to Mr. A. C. Scibiorski.

Professor Mabel I. Nowlan, on leave from Bethel Woman's College of Kentucky, is assisting with the work of the War Department.

Associate Professor C. A. Rupp of Pennsylvania State College is now a captain in the Signal Corps in Washington.

The following appointment to an instructorship is announced:
Denison University (Pre-Meteorological Program): Paul Cramer

Professor Emeritus Lilian Hackney of Marshall College died on February 4, 1943.

The death is announced of Professor David Hilbert of the University of Göttingen. He was eighty-one years old.

Professor H. F. Minssen, head of the mathematics department and vice-president of San Jose State College, died on February 7, 1943.

Professor H. H. Mitchell of the University of Pennsylvania died on March 13, 1943.

Professor M. H. Tyler of Rhode Island State College died on December 16, 1942.

Professor J. J. Westemeier of Dowling College (Des Moines, Iowa) died on February 1, 1943.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1943:

Brown University. Semester I of the three semester academic year for regular students will begin about July 1 and extend until about October 23. Essentially no formal courses in pure mathematics will be offered. However, the Program of Advanced Instruction and Research in Mechanics enters on its third year on June 14 and will extend to September 4. Courses in the following subjects are planned: partial differential equations, advanced theoretical mechanics, theory of structures, elasticity, fluid dynamics, differential and integral equations of physics, practical analysis, mathematical optics. The faculty includes Professors Stefan Bergman, L. N. Brillouin, Willy Feller, Witold Hurewicz, P. W. Ketchum, Willy Prager, J. D. Tamarkin and S. P. Timoshenko.

The University of California. From *July 5 to October 23* the following graduate courses will be offered. By Professor McDonald: theory of functions of a complex variable. By Professor Buck: differential equations. By Professor Sperry: differential geometry. By Professor Foster: algebra. By Professor Neyman: probability. By Mr. Eudey: statistical problems in experimentation. By Professors Bernstein, Foster and Tarski: seminar on topics in higher algebra. By Professor McDonald: seminar on function geometry and complex variable. By Professor Evans: seminar on elasticity, wave theory, shock waves, characteristics. By Professor Lehmer: seminar on advanced computation for pure and applied

mathematics. By Professor Neyman: seminar on probability and statistics. From *June 28 to August 7* the following courses will be offered. By Professor Tarski: famous problems of elementary algebra. By Dr. Wakerling: survey of mathematics.

The Catholic University of America. In addition to the elementary courses the following will be offered: *First term, June 28 to August 7.* By Professor Rice: analytic geometry of three dimensions. By Professor Finan: modern algebraic theories. By Professor Ramler: college geometry, differential equations, analytic projective geometry. *Second term, August 9 to September 18.* By Professor Rice: numerical mathematical analysis. By Professor Finan: abstract algebra. By Professor Ramler: advanced analytic geometry.

The University of Chicago. In addition to courses in trigonometry, college algebra, analytical geometry and calculus, theory of equations and differential equations, the following will be offered beginning June 22: By Professor Albert: higher algebra. By Professor Logsdon: analytic projective geometry. By Professor Graves: functions of a complex variable. By Professor Schilling: applications of differential equations to physics.

Columbia University. The following graduate courses will be given from July 7 to August 13: By Professor Kasner: general introduction to modern mathematics, transformations and curves. By Professor Koopman: probability. By Professor Murray: differential equations. By Professor Ritt: functions of a real variable.

The University of Illinois. From June 14 to August 7 graduate courses will be offered in the following subjects: theory of statistics, differential equations of mathematical physics, algebra, analysis, geometry. From June 14 to October 2 the following intermediate courses will be offered: differential equations, advanced calculus, statistics, fundamental concepts, introduction to higher algebra, introduction to higher analysis.

The State University of Iowa. In addition to the undergraduate courses the following will be offered over the period from June 5 to July 30: By Dr. Price: studies in high school mathematics. By Dr. Berg: differential equations. By Professor Knowler: elements of statistics. By Professor Chittenden: advanced calculus (definite integrals), seminar in analysis. By Professor Wylie: applied astronomy. By Professor Woods: solid analytic geometry, constructive geometry. By Professor Conkwright: group theory, navigation and maps. From August 2 to August 20 there is an independent study unit for graduate students.

The University of Michigan. In addition to elementary courses and courses in differential equations, theory of equations and determinants, advanced calculus, the following courses will be offered in the period from June 28 to August 20: By Professor Anning: modern geometry. By Professor Carver: introduction to air navigation. By Professor Copeland: statistics. By Professor Craig: theory of statistics I and II. By Professor Dwyer: computational methods. By Professor Hildebrandt: real variables, algebraic theory. By Professor Karpinski: teaching of geometry, history of arithmetic and algebra. By Professor Nesbitt: theory

of probability. By Professor Rainich: topics in higher geometry. By Professor Steenrod: introduction to the theory of sets. By Professor Wilder: introduction to the foundations of mathematics. In addition there will be a seminar in pure mathematics and one in statistics conducted by Professor Craig.

The University of North Carolina. In addition to the elementary courses, the following will be offered in the period from June 10 to August 27: By Professor Browne: theory of equations. By Professor Garner: the history and teaching of mathematics. By Professor Henderson: differential equations. By Professor Hobbs: partial differential equations. By Professor Trimble: mechanical drawing. By Professor Winsor: college geometry. By Mr. E. T. Hodges, Lieut., U. S. Coast Guard (Ret.): marine navigation. By Dr. Seebeck: aerial navigation.

The Ohio State University. In addition to the elementary courses the following will be offered in the period from June 21 to September 3: By Professor Bamforth: partial differential equations. By Professor Wylie: advanced mathematics for engineering students, fundamental ideas in geometry. By Dr. Albert: advanced calculus. By Dr. Mickle: theory of equations. By Dr. Helsel: differential equations.

The University of Texas. The following advanced courses will be offered. *First term, June 1 to July 22.* By Professor R. L. Moore: introduction to the foundations of geometry. By Professor H. J. Ettlinger: differential equations, selected topics in mathematical physics. By Professor E. F. Beckenbach: analytic functions. By Professor P. M. Batchelder: advanced calculus. By Professor H. V. Craig: vector and tensor analysis. By Professor E. W. Titt: advanced applied mathematics. By Professor R. N. Haskell: dynamics. By Dr. N. Coburn: meteorology. *Second term, July 23 to September 11.* By Professor H. J. Ettlinger: advanced calculus, descriptive geometry. By Professor R. G. Lubben: non-Euclidean geometry. By Professor A. E. Cooper: theory of functions of a complex variable. By Professor E. W. Titt: dynamics. By Dr. N. Coburn: meteorology.

The University of Wisconsin. From June 18 to September 18 the following advanced courses will be offered: By Professor Cohen: higher mathematics for engineers. By Professor Ulam: differential equations. During the six-weeks session the following courses will be given: By Professor Kenney: theory of equations, mathematics of educational statistics. By Mrs. Sokolnikoff: advanced calculus, mathematical applications. By Professor Trump: college geometry.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE MINNESOTA PROGRAM IN PRE-FLIGHT MATHEMATICS

In September, 1942, a committee was organized in Minneapolis, Minnesota, to foster free instruction in mathematics for men about to become aviation cadets in the Army or the Navy. Courses were outlined, and the plan was pushed vigorously all over Minnesota by the Committee on Education of the Minnesota Civilian Defense Council.

To inaugurate the program, the names of boys who were eligible for the training were obtained from the Aviation Cadet Examining Board. A card and a letter were then sent to each boy; the card giving pertinent information upon the boy's training and upon his interest in the proposed course in mathematics was to be returned. Shortly thereafter, a letter was sent to all the superintendents of schools throughout the state, and to local educational representatives on the Civilian Defense Council. The letter contained a list of the boys in the community who were interested in the course in mathematics, and asked for assistance in finding volunteers who would serve as instructors. Almost without exception, favorable replies were received, and outlines of the courses were then sent to the persons volunteering for the teaching. When the project was well under way, ninety-five towns in the state were giving instruction without charge to the men enrolled.

The courses in mathematics were designed to provide as much review in the fundamentals of arithmetic, algebra, and plane geometry as each student needs, and then to teach as much new material, including trigonometry, as is necessary to solve standard problems in aviation. Outlines of Courses A and B used in the mathematics instruction may be obtained by addressing The Office of Civilian Defense, State Capitol Building, St. Paul, Minnesota.

Minneapolis started its courses October 1, 1942, and thus became the first community in the state to have the project in operation. The instruction was done entirely by volunteers; in fact, the staff at the start included three retired teachers, three housewives, three members of the faculty of the University of Minnesota, one industrial chemist, one public school teacher, two secretaries and one life underwriter. Classes met six hours each week. Enrollment was limited to men who had enlisted in the Air Corps of either the Army or the Navy. Students were assigned to classes on the basis of their previous training in mathematics, and within a month after the start of the program, nine classes were in operation with a total enrollment of 334. The project originally was intended to extend over a period of six weeks, and to be repeated every six weeks as new enlistments occurred. It was found, however, that a more fluid organiza-

tion was desirable. New enlistees, therefore, were admitted as soon as they applied, without requiring them to wait for the opening of a new term. At the end of a period of six weeks, a reassignment of students was made within the school. Under this plan, the students progressed from class to class, and continued in classes until they were called into service, thus acquiring the maximum amount of preparation for their future needs at the aviation training centers.

The foregoing project was an outgrowth of the experience of other agencies with pre-flight courses. As early as July, 1941, the Minneapolis Public Schools conducted review courses for boys who expected to take the qualifying examinations for the Army Air Corps. These courses were discontinued in January, 1942, when the qualifying examinations in specific subjects were replaced by a screening test. Then during the summer months of 1942, the Extension Service of the University of Minnesota gave courses in mathematics to nearly five hundred boys who were awaiting assignment to the schools of the Air Corps. Professor W. L. Hart became very much interested in this training, and advocated the idea that boys should not be required to pay for this service. Accordingly, he and Dr. Clifford Archer of the College of Education of the University of Minnesota formulated the preliminary proposals for the present program.

A BIBLIOGRAPHY OF CRYPTOGRAPHY

The following bibliography on cryptography was compiled from several sources; in particular, thanks are due Lieutenant Commander H. T. Engstrom of the United States Navy for his assistance. Titles in the list which are preceded by asterisks are known to be available at the present time.

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Paul E. Valerio, *De la cryptographie; essai sur les methodes de dechiffrement*, Librairie militaire de L. Baudoin, 1893–96.

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The American Cryptogram Association publishes a bi-monthly journal, *The Cryptogram*. For information upon the association and its publications, communicate with Mr. George C. Lamb, Burton, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
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